Problem # 4.
a) Examples show that a system of 4 linear equations in 3 variables can have no solutions, a unique solution, or infinitely many solutions. Consider, for example, systems

\[
\begin{align*}
    x_1 &= 1 \\
    x_2 &= 1 \\
    x_3 &= 1 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 &= 1 \\
    x_2 &= 1 \\
    x_3 &= 1,
\end{align*}
\]

and

\[
\begin{align*}
    x_1 &= 1 + x_2 + x_3 = 1 \\
    2x_1 + 2x_2 + 2x_3 &= 2 \\
    3x_1 + 3x_2 + 3x_3 + 3x_4 &= 3 \quad \text{and} \quad
\end{align*}
\]

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 0 \\
    x_1 + x_2 + x_3 &= 3 \\
    4x_1 + 4x_2 + 4x_3 &= 4.
\end{align*}
\]

b) A system of 3 linear equations in 4 variables cannot have a unique solution, since once we reduce the system to the reduced row-echelon form, there has to be at least one free variable. The following examples show that such a system can have no solutions or infinitely many solutions:

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 1 \\
    2x_1 + 2x_2 + 2x_3 + 2x_4 &= 1 \quad \text{and} \quad
\end{align*}
\]

\[
\begin{align*}
    2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2 \\
    3x_1 + 3x_2 + 3x_3 + 3x_4 &= 3
\end{align*}
\]

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 1 \\
    2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2
\end{align*}
\]

\[
\begin{align*}
    3x_1 + 3x_2 + 3x_3 + 3x_4 &= 3
\end{align*}
\]

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 0 \\
    x_1 + x_2 + x_3 &= 3
\end{align*}
\]

\[
\begin{align*}
    4x_1 + 4x_2 + 4x_3 &= 4.
\end{align*}
\]

c) Let \( \bar{x} \) be a solution of the system \( C \bar{x} = \bar{c} \) for some vector \( \bar{c} \) and \( C = AB \). Let \( B \bar{x} = \bar{y} \), so \( A \bar{y} = \bar{c} \). Since \( B \) is a 3 \( \times \) 4 matrix, by Part b) there exists infinitely many vectors \( \bar{z} \) such that \( B \bar{z} = \bar{y} \). Then, for every such a vector \( \bar{z} \), we have \( C \bar{z} = (AB) \bar{z} = A(B \bar{z}) = A \bar{y} = \bar{c} \). This proves that the system \( C \bar{x} = \bar{c} \) cannot have a unique solution. The system can have no or infinitely many solutions as the following examples show:

\[
\begin{align*}
    \text{let } A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so } C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\end{align*}
\]

Consider the systems:

\[
\begin{align*}
    x_1 &= 1 \\
    x_2 &= 1 \\
    x_3 &= 1 \\
\end{align*}
\]

and

\[
\begin{align*}
    0x_1 + 0x_2 + 0x_3 + 0x_4 &= 1 \\
    0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0.
\end{align*}
\]

Answer: a) Yes, Yes, Yes; b) Yes, No, Yes; c) Yes, No, Yes.