Problem # 3.

a) Let us choose some linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbf{R}^3$ and the linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that $T(\vec{x}) = \vec{0}$ for all $\vec{x}$ in $\mathbf{R}^3$. Then $T(\vec{a})$, $T(\vec{b})$, $T(\vec{c})$ are not linearly independent. **Answer:** False.

b) Suppose that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ for some numbers x, y, z. Applying T to both sides of the equation, we get

$$\vec{0} = T(\vec{0}) = T(x\vec{a} + y\vec{b} + z\vec{c}) = xT(\vec{a}) + yT(\vec{b}) + zT(\vec{c}),$$

from which we must have $x = y = z = 0$, since $T(\vec{a})$, $T(\vec{b})$, and $T(\vec{c})$ are linearly independent. Therefore, $\vec{a}, \vec{b}$, and $\vec{c}$ are linearly independent. **Answer:** True.

c) Let us choose some three vectors $\vec{a}, \vec{b}, \vec{c}$ that span $\mathbf{R}^3$ and the linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that $T(\vec{x}) = \vec{0}$ for all $\vec{x}$ in $\mathbf{R}^3$. Then $T(\vec{a})$, $T(\vec{b})$, $T(\vec{c})$ do not span $\mathbf{R}^3$. **Answer:** False.

d) Since the three vectors $T(\vec{a})$, $T(\vec{b})$, $T(\vec{c})$ span $\mathbf{R}^3$ and the dimension of $\mathbf{R}^3$ is precisely 3, the vectors $T(\vec{a})$, $T(\vec{b})$, $T(\vec{c})$ must be linearly independent. Then, by Part b), the three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent. Since the dimension of $\mathbf{R}^3$ is precisely 3, the vectors $\vec{a}, \vec{b}, \vec{c}$ span $\mathbf{R}^3$. **Answer:** True.