1. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Find all eigenvalues and all eigenvectors of A.

Solution. To find the eigenvalues, we use the shortcut. The sum of the eigenvalues is the trace of A, that is, $1 + 4 = 5$. The product of the eigenvalues is the determinant of A, that is, $1 \cdot 4 - (-1) \cdot 2 = 6$, from which the eigenvalues are 2 and 3.

Next, we find the eigenvectors. The eigenvectors with the eigenvalue 2 are the non-zero vectors $\vec{x}$ satisfying the equation $A\vec{x} = 2\vec{x}$, that is, $(A - 2I)\vec{x} = \vec{0}$, that is,

$$\begin{bmatrix} \underline{x_1} & \underline{x_2} & & \\ -1 & -1 & \mid & 0 \\ 2 & 2 & \mid & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{x_1} & \underline{x_2} & & \\ 1 & 1 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix},$$

from which the non-zero solutions are

$$\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{for any} \quad x_2 \neq 0.$$

Similarly, the eigenvectors with the eigenvalue 3 are the non-zero vectors $\vec{x}$ satisfying the equation $A\vec{x} = 3\vec{x}$, that is, $(A - 3I)\vec{x} = \vec{0}$, that is,

$$\begin{bmatrix} \underline{x_1} & \underline{x_2} & & \\ -2 & -1 & \mid & 0 \\ 2 & 1 & \mid & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{x_1} & \underline{x_2} & & \\ 1 & 1/2 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix},$$

from which the non-zero solutions are

$$\begin{bmatrix} -1/2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, \quad \text{for any} \quad x_2 \neq 0.$$

Answer. The eigenvalues are 2 and 3. The eigenvectors with the eigenvalue 2 are the vectors $t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where t can be any non-zero number. The eigenvectors with the eigenvalue 3 are the vectors $t \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where t can be any non-zero number.