1. Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$. Find all matrices X such that $AX = XA$.

Solution. Letting $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad \text{that is,}$$

$$\begin{bmatrix} a+3c & b+3d \\ 3a+2c & 3b+2d \end{bmatrix} = \begin{bmatrix} a+3b & 3a+2b \\ c+3d & 3c+2d \end{bmatrix}.$$

Hence we get the system of linear equations

$$\begin{aligned} a+3c &= a+3b \\ b+3d &= 3a+2b \\ 3a+2c &= c+3d \\ 3b+2d &= 3c+2d \end{aligned}, \quad \text{or, in the matrix form,}$$

$$\begin{bmatrix} \underline{a} & \underline{b} & \underline{c} & \underline{d} & & \\ 0 & -3 & 3 & 0 & | & 0 \\ -3 & -1 & 0 & 3 & | & 0 \\ 3 & 0 & 1 & -3 & | & 0 \\ 0 & 3 & -3 & 0 & | & 0 \end{bmatrix}.$$

Solving the system (computations omitted), we get $d = s$, $c = t$, $b = t$, $a = s - t/3$, where s and t can be any numbers.

Answer. $X = \begin{bmatrix} s - t/3 & t \\ t & s \end{bmatrix}$, where s and t can be any numbers.