1. Consider a linear transformation with the matrix \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} \).

Find a basis of the kernel and a basis of the image of the transformation. Describe the kernel and image geometrically (as a line, plane, etc.).

**Solution.** Reducing the matrix, we get

\[
\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The kernel of \( A \) consists of all solutions of the system \( A\vec{x} = \vec{0} \), that is, of the vectors

\[
\begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},
\]

where \( x_3 \) can be any number. Hence \( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \) is a basis of the kernel and the kernel is a line in \( \mathbb{R}^3 \). This copy of \( \mathbb{R}^3 \) is the input space.

To pick a basis of the image, we select the columns of \( A \) corresponding to the leading 1s. Thus the vectors \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \) is a basis of the image of \( A \). Hence the image is a plane in \( \mathbb{R}^3 \). This copy of \( \mathbb{R}^3 \) is the output space.

**Answer.** The kernel is a line in \( \mathbb{R}^3 \) (input space) with a basis consisting of the vector \( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \). The image is a plane in \( \mathbb{R}^3 \) (output space) with a basis consisting of the vectors \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \).