2. Let $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ be linearly independent vectors from $\mathbf{R}^n$. Are the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$ necessarily linearly independent?

Solution. To check whether the vectors are linearly independent, we must answer the following question: if a linear combination of the vectors is the zero vector, is it necessarily true that all the coefficients are zeros?

Suppose that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$$

(a linear combination of the vectors is the zero vector). Is it necessarily true that $x_1 = x_2 = x_3 = 0$?

We have

$$\begin{aligned} &x_1\vec{v}_1 + x_2\vec{v}_2 + x_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_1 + x_3\vec{v}_2 + x_3\vec{v}_3 \\ =&(x_1 + x_3)\vec{v}_1 + (x_2 + x_3)\vec{v}_2 + x_3\vec{v}_3 = \vec{0}. \end{aligned}$$

Since $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent, we must have the coefficients of the linear combination equal to 0, that is, we must have

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0\,, \\ x_3 &= 0 \end{aligned}$$

from which it follows that we must have $x_1 = x_2 = x_3 = 0$. Hence the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly independent.

Answer. The vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly independent.