2. Let \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) be linearly independent vectors from \( \mathbb{R}^n \). Are the vectors \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \) necessarily linearly independent?

Solution. To check whether the vectors are linearly independent, we must answer the following question: if a linear combination of the vectors is the zero vector, is it necessarily true that all the coefficients are zeros?

Suppose that

\[
x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 (\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}
\]

(a linear combination of the vectors is the zero vector). Is it necessarily true that \( x_1 = x_2 = x_3 = 0 \)?

We have

\[
x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 (\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_1 + x_3 \vec{v}_2 + x_3 \vec{v}_3 = (x_1 + x_3) \vec{v}_1 + (x_2 + x_3) \vec{v}_2 + x_3 \vec{v}_3 = \vec{0}.
\]

Since \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) are linearly independent, we must have the coefficients of the linear combination equal to 0, that is, we must have

\[
x_1 + x_3 = 0
\]

\[
x_2 + x_3 = 0,
\]

\[
x_3 = 0
\]

from which it follows that we must have \( x_1 = x_2 = x_3 = 0 \). Hence the vectors \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \) are linearly independent.

Answer. The vectors \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \) are linearly independent.