

Buildings and their applications in geometry and topology

Lizhen Ji*

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Abstract

In this paper, we briefly introduce different types of buildings such as spherical buildings, Euclidean buildings, twin buildings, \mathbb{R} -buildings and describe some of their applications to many subjects: (1) differential geometry such as Mostow strong rigidity, rank rigidity of nonpositively curved manifolds, Margulis superrigidity, quasi-isometry rigidity, classification of isoparametric manifolds, compactifications of symmetric spaces and locally symmetric spaces, (2) topology such as cohomology and duality properties of arithmetic groups, simplicial volume of locally symmetric spaces and Novikov conjectures, (3) analysis such as harmonic maps, harmonic analysis and representation theory of p -adic Lie groups, (4) algebra such as finite simple groups, infinite simple groups, algebraic K -theory and representations of algebraic groups over finite fields. By putting together many different types of buildings and applications in seemingly unrelated topics, we hope to present an overview of rich structures and applications of buildings and the underlying groups.

Keywords. Tits building, Bruhat-Tits building, Bruhat-Tits theory, spherical building, Euclidean, topological building, twin building, arithmetic group, Mostow strong rigidity, Margulis super-rigidity, BN -pair, compactification, quasi-isometry rigidity, Kac-Moody group, $CAT(0)$ -space, $CAT(0)$ -group, Solomon-Tits Theorem, cohomology group, cohomology dimension, generalized Poincaré duality, simplicial volume, Coxeter complex, incidence geometry, finite geometry, simplicial complex, tree, geometrization conjecture of Thurston, curve complex, Teichmüller space, mapping class group.

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*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109.
Email address, lji@umich.edu

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1 Introduction and history of buildings

1.1 Summary

Classical simple Lie groups over \mathbb{C} are the symmetry groups of quadratic forms or sesquilinear forms of finite dimensional vector spaces over \mathbb{C} . The same construction works for vector spaces over finite fields and produces classical finite groups of Lie type, but this method does not extend to exceptional simple Lie groups. Buildings were originally introduced by Tits in 1950s in order to realize exceptional Lie groups as the symmetry groups of spaces (or geometry) so that one can construct geometrically analogues of exceptional simple Lie groups over arbitrary fields.¹

The first buildings were spherical buildings, and they have been generalized in various ways: Euclidean buildings (Bruhat-Tits buildings), topological buildings, \mathbb{R} -buildings, in particular \mathbb{R} -trees, twin buildings and hyperbolic buildings. Besides the combinatorial properties, they often carry additional structures from geometry and group theory. They are useful for many different applications in various subjects such as algebraic groups, group theory and in particular finite groups, finite geometry, representation theory over local fields, algebraic geometry, Arakelov

¹According to [RT, p. 292], "... it is perhaps worth remarking that one of the initial motivations for the theory of buildings, at a time when Chevalley's fundamental "Tohoku paper" had not yet appeared, was the search for a geometric way of obtaining algebraic analogues of the exceptional Lie groups."

intersection for arithmetic varieties, algebraic K -theories, combinatorial and geometric group theory, global differential geometry, geometric and algebraic topology, cohomology groups of arithmetic groups and S -arithmetic groups, Mostow strong rigidity and Margulis super-rigidity of lattice subgroups of semisimple Lie groups and of nonpositively curved manifolds, classification of isoparametric submanifolds in \mathbb{R}^n of high codimension, and existence of hyperbolic structures on three dimensional manifolds in Thurston's geometrization program.

Basically, there are two reasons why buildings are useful for questions and problems on semisimple (or more generally reductive) groups and their subgroups, and generalizations such as Kac-Moody groups:

1. For semisimple (or reductive) Lie groups and algebraic groups, buildings provide effective ways to parametrize subgroups such as parabolic subgroups of reductive algebraic groups over any fields and parahoric subgroups of reductive algebraic groups over locally compact, totally disconnected fields, and relations between these subgroups can also be expressed and reflected effectively by the geometry of buildings. Hence properties and structures of the groups can be described, understood, and proved via buildings. Rich geometry (for example existence of ample collections of subspaces) of buildings is a geometric reflection of rich structure of the associated (parahoric) groups.
2. Buildings also provide spaces on which the algebraic groups and their subgroups act, and representation spaces of these groups can be constructed and studied. Furthermore, the automorphism groups of locally finite buildings provide natural locally compact topological groups which can be used to understand groups that act on the buildings, for example, the Kac-Moody groups.

In this paper, several applications of buildings in differential geometry, geometric topology and group cohomology theory will be emphasized. There are four underlying themes in these applications:

1. Spherical buildings often describe the geometry at infinity of symmetric spaces and locally symmetric spaces, and hence are large scale geometric invariants of these spaces. The topology of spherical buildings often describes precisely the topology at infinity of locally symmetric spaces. Both spherical buildings and \mathbb{R} -buildings also appear as limiting objects under degeneration or scaling of Riemannian manifolds and more generally metric spaces.
2. Euclidean buildings are analogues of symmetric spaces for semisimple groups defined over local fields and their discrete subgroups, and they play similar roles in understanding properties and structures of algebraic groups and their subgroups, for example, generalization of the standard decompositions such as the Cartan decomposition and Iwasawa decomposition.
3. Euclidean buildings and symmetric spaces of noncompact type provide the most important examples of $CAT(0)$ -spaces, which are simply connected and

nonpositively curved geodesic metric spaces. This fact is used crucially in proving the super-rigidity and arithmeticity of co-finite discrete subgroups of the rank one simple Lie group $Sp(1, n)$ and $F_{4(-20)}$, the isometry groups of quaternionic hyperbolic space and the Cayley plane. Furthermore, groups that act properly and isometrically on buildings provide some of the most interesting groups in geometric group theory, for example, Kac-Moody groups over finite fields.

4. Spherical buildings of higher rank have large symmetry groups and are rigid, and hence objects which contain or induce higher rank buildings tend to be rigid.
5. Additional structures on buildings, for example, topological buildings, are important in applications for infinite groups and nonpositively curved manifolds such as rigidity properties of lattices of Lie groups and the rank rigidity of nonpositively curved compact Riemannian manifolds.

We will provide more details and references to support these themes and give more precise statements of the vague results mentioned here in the rest of this paper.

1.2 History of buildings and outline of the paper

In some sense, the theory of buildings is one of the best examples to illustrate the power of the celebrated Erlangen program by Klein and the geometry envisioned by Lie.

In 1872, at the tender age of 23, Felix Klein assumed his professorship and delivered his inaugural address at the University of Erlangen with a title *Comparative review of recent results in geometry* [Kle]². In the talk, he proposed a program to understand geometry from the point of view of groups and hence to unify and classify different geometries. This became known as the famous Erlangen program, which was summarized in one sentence by Klein: *Geometry is the science which studies the properties of figures preserved under the transformation of a certain group of transformations, or, as one also says, the science which studies the invariants of a group of transformations*. The Erlangen program has had profound impact in the development of mathematics. In fact, the philosophy of finding and understanding invariants of automorphism groups of any system or structure can be found in many branches of mathematics, for example, homotopy invariants and topological (or homeomorphic) invariants in algebraic topology, birational invariants in algebraic geometry, conformal invariants in analysis, and isometric invariants in geometry, and quasi-isometric invariants in large scale (coarse) geometry. This point of view has also been very useful in physics and other sciences.

There are two implications in the Erlangen program:

²This paper is also available on the arXiv, <http://arxiv.org/pdf/0807.3161v1>

1. Given a geometry, for example, the Euclidean geometry, the hyperbolic geometry or the projective geometry, find the transformation group which preserves properties of figures and configurations of subspaces in this geometry.³ The purpose of geometry is to understand quantities and properties of these subspaces and their relations which are invariant under action of the symmetry group.
2. Conversely, given a group, find a geometry with a rich collection of subspaces such that the group acts on it and preserves its geometric properties, and to realize the group as the automorphism group of the geometry.

In general, we expect the group to be sufficiently large, for example, the action is transitive and the space becomes a homogeneous space of the group, or more generally, the quotient of the space by the group is compact, i.e., the action is transitive up to a compact subset. In the Erlangen program, it is not only the total transformation group which is important. Subgroups, such as stabilizers of points or figures (or configurations) are integral parts of the description of geometry in terms of groups. For example, different homogeneous spaces of a Lie group are determined by the stabilizers of points in the spaces. In the projective geometry, the incidence relation between points, lines and subspaces in projective spaces can be described in terms of the inclusion relation of their stabilizers in the projective linear group.

Besides the stabilizer subgroups of points and figures (or configurations), other subgroups are also important, for example, subgroups which preserve some additional structures on the space. In fact, Klein obtained the Euclidean geometry and hyperbolic geometry as specializations of the projective geometry by using suitable subgroups of the projective linear group (see [Kli] for more detail).

It is known that the family of simple complex Lie algebras (or groups) consists of

1. four classical infinite series: (A_n) , $\mathfrak{sl}(n+1, \mathbb{C})$; (B_n) , $\mathfrak{so}(2n+1, \mathbb{C})$; (C_n) , $\mathfrak{sp}(n, \mathbb{C})$; (D_n) , $\mathfrak{so}(2n, \mathbb{C})$,
2. and five exceptional ones: E_6 , E_7 , E_8 , F_4 and G_2 .

The classical ones correspond to classical (metrical) geometry in the sense that their compact real forms of the complex Lie groups $SU(n)$, $SO(n)$ and $Sp(n)$ correspond to the identity component of the isometry groups of the projective spaces $\mathbb{C}P^{n-1}$, $\mathbb{R}P^{n-1}$ and $\mathbb{H}P^{n-1}$, where \mathbb{H} is the algebra of quaternions.

The exceptional simple complex Lie groups correspond to the automorphism groups of the projective planes over the algebras \mathbb{O} , $\mathbb{O} \otimes \mathbb{C}$, $\mathbb{O} \otimes \mathbb{H}$ and $\mathbb{O} \otimes \mathbb{O}$, where \mathbb{O} is the algebra of octonions (or Cayley numbers). But due to the non-associativity of \mathbb{O} , these spaces do not satisfy the usual axioms for a projective

³According to wikipedia, geometry is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space. For interesting geometries, the space should contain a rich collection of subspaces and their configurations, and it should also admit a large symmetry group which acts on these subspaces and their configurations. Any choice of subspaces and relations between them determine the geometry of underlying space.

plane, and some other geometric models are needed for the exceptional Lie groups (see [Bae] for a nice summary of the geometry of these spaces and relations to the exceptional groups).

To carry out the Erlangen program for the exceptional groups and also to realize exceptional groups over other fields, for example over finite fields in order to get finite groups of exceptional Lie types, Tits introduced spherical buildings. It should probably be emphasized that in the theory of Tits buildings (or Tits geometry), it is the incidence geometry but not the metric geometry which is emphasized. In the incidence geometry, a crucial concept is the incidence relation which is the inclusion relation between distinguished subspaces [Bue]. (Note that the incidence relation between points, lines and other linear subspaces is a crucial part of the projective geometry over \mathbb{R} and \mathbb{C}).

Spherical Tits buildings were first introduced by Tits in a series of papers from 1950s on (see [Ti4] [Ti5] [Ti6]), which resulted in the fundamental book [Ti2]. According to [Ti2, p. v], “The origin of the notions of buildings and BN-pairs lies in an attempt to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups”. Once such geometric interpretation of groups as automorphism groups of buildings is achieved and buildings are constructed, then exceptional groups over any fields can be constructed and understood geometrically [RT].⁴

Spherical buildings are simplicial complexes which contain a “large” collection of distinguished sub-complexes called apartments, which are finite triangulations of spheres and satisfy suitable compatibility and symmetry conditions (see §2 below for more details). For any semisimple linear algebraic group \mathbf{G} defined over any field k , there is a spherical building $\Delta(\mathbf{G})$ whose simplexes correspond to (proper) parabolic subgroups of \mathbf{G} defined over k . Let r be the k -rank of \mathbf{G} , i.e., the maximal dimension of k -split tori contained in \mathbf{G} . Then $\Delta(\mathbf{G})$ is a simplicial complex of dimension $r - 1$ and is called a building of rank r . (Note that the rank of a spherical building is equal to the rank of the algebraic group and also equal to the number of vertices of a maximal dimensional simplex of the building.) If the field k is finite, the building is a finite simplicial complex. Otherwise, it is infinite in general. Clearly, $\mathbf{G}(k)$ acts on the set of parabolic subgroups defined over k and hence acts on $\Delta(\mathbf{G})$ by simplicial automorphisms. It turns out that the group $\mathbf{G}(k)$ acts transitively on $\Delta(\mathbf{G})$ (in fact, a stronger version of transitivity holds). The rich geometry of $\Delta(G)$ reflects group theoretic properties of G . For example, the existence and compatibility conditions on apartments of $\Delta(G)$ is equivalent to the Bruhat decomposition of G with respect to a minimal parabolic subgroup of G . This gives the first implication in the Erlangen program: *Constructing geometries from groups*.

On the other hand, given a (spherical) building Δ , its automorphism group provides interesting examples of groups. Once buildings can be constructed independently from groups, it gives a geometric method to construct groups, in

⁴This geometric approach to construct exceptional groups over finite fields did not seem to be the driving force for buildings anymore after the fundamental work of Chevalley [Che], where for any simple complex Lie algebra, an integral basis is constructed, and a corresponding Lie algebra and hence algebraic group over any field can be defined.

particular to construct exceptional algebraic groups over finite fields (see for example [RT] [Ro1]), which was the original motivation of Tits as mentioned above. An important result of Tits [Ti2]⁵ shows that a thick spherical building of rank at least 3 arises essentially from a semisimple algebraic group and in this case the group is completely determined by the building. This shows that the group is described by the geometry and gives the second implication of the Erlangen program: *Constructing groups from geometries*.

An effective way to construct buildings for a group G is to use a BN-pair, a pair of subgroups B and N of G which satisfy certain conditions. Such pairs are often called a *Tits system*. This also shows that buildings give geometric ways to describe properties of G , its subgroups and relations between them.

In [IM], Iwahori and Matsumoto introduced a new type of BN-pairs in Chevalley groups \mathbf{G} defined over a field complete with respect to a discrete valuation, for example, a non-archimedean local field k , i.e., a locally compact field complete with respect to a discrete valuation. As a consequence, conjugacy classes of maximal compact subgroups of $\mathbf{G}(k)$ are determined, and explicit description of the Hecke ring of $\mathbf{G}(k)$ with respect to maximal compact subgroups is obtained. The BN-pairs in [IM] started a new type of building in which an apartment is a triangulation of a Euclidean space. These buildings are called Euclidean buildings. Later, Bruhat and Tits [BT1] constructed a Euclidean building $\Delta^{BT}(\mathbf{G})$ for a linear (simply connected) semisimple (or reductive) algebraic group \mathbf{G} defined over a non-archimedean local field k in order to understand structures of subgroups, for example, compact open subgroups, of $\mathbf{G}(k)$. Due to this result, Euclidean buildings are often called Bruhat-Tits buildings. The dimension of the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is equal to r , the k -rank of the algebraic group \mathbf{G} . When the rank r is equal to 1, the building $\Delta^{BT}(\mathbf{G})$ becomes a tree. The Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ and the spherical Tits building $\Delta(\mathbf{G})$ are closely related. In fact, the latter appears at the infinity of the former.

The primary properties of buildings are of combinatorial nature, for example, the incidence relations between simplexes. They can be enhanced in different ways. For example, by considering algebraic subgroups or algebraic subgroup schemes associated with vertices and facets of Bruhat-Tits buildings, we obtain the Bruhat-Tits theory.

Remark 1.2.1. It might be worthwhile to point out that the Bruhat-Tits buildings is only one part of the vast Bruhat-Tits theory as developed by Bruhat and Tits [BT1] [BT2] [BT3] [BT4]. It seems that the main point of the Bruhat-Tits theory is to understand structures of and relations between subgroups of reductive algebraic groups $\mathbf{G}(k)$ over a non-archimedean local field k such as parahoric subgroups and compact open subgroups. The underlying Bruhat-Tits buildings give an efficient and visually effective way to parametrize these subgroups and to clarify relations between them, and the global geometric properties of the Bruhat-Tits buildings such as being a CAT(0)-space (i.e., being a simply connected geodesic

⁵More precisely, it arises from a semisimple algebraic group or a classical group or a group of type F_4 defined over a pair of fields F and K in characteristic 2 such that $K^2 \subset F \subset K$. See 30.33 in [Weis1] for more details.

metric space with nonpositive curvature) has also been used crucially and in a profound way to understand structures of subgroups of $\mathbf{G}(k)$ and representation theory of $\mathbf{G}(k)$. To show that the Bruhat-Tits buildings have the desired rich geometry such as existence of apartments, chambers, and a large symmetry group also depends heavily on the structures of the algebraic groups \mathbf{G} as integral group schemes. In this sense, both the underlying topological and geometric aspects of the Bruhat-Tits buildings and the group theoretic aspects of the Bruhat-Tits theory are closely intertwined. This is especially important for problems about harmonic analysis and representation theories of $\mathbf{G}(k)$. On the other hand, for many questions in geometry and topology, what is needed and used is the underlying simplicial complexes, i.e., the Bruhat-Tits buildings, their geometric properties, and actions of discrete groups on them, without involving the fine points in the Bruhat-Tits theory. The latter point of view seems to be reflected by the contents of the current existing books on buildings listed in the references such as [AbB] [Br1] [Garr] [Ro2].

Another more direct geometric enhancement of a building is to define a geodesic metric on the building that is invariant under the group. More precisely, since apartments can be endowed with metrics which are compatible on their intersection, these metrics can be glued together into a canonical geodesic metric, called the *Tits metric*, on the building. With respect to the topology defined by this metric, Euclidean buildings are noncompact. In fact, they are CAT(0)-spaces (see Proposition 4.1.3 below). It turns out that locally compact Euclidean buildings can be compactified by adding at infinity suitable spherical Tits buildings which are given a topology different from the one induced by its canonical Tits metric [BoS1]. When \mathbf{G} is a linear semisimple algebraic group defined over a non-archimedean local field k , then the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is locally compact, and the spherical Tits building which compactifies $\Delta^{BT}(\mathbf{G})$ is exactly the building $\Delta(\mathbf{G})$ with a *suitable topology*, which is in general different from the Tits metric topology. (As will be seen below, the two buildings $\Delta(\mathbf{G})$ and $\Delta^{BT}(\mathbf{G})$ arise from two different choices of BN -pairs in $\mathbf{G}(k)$, or rather two different choices of B but the same N).

We note that with respect to the topology induced from the Tits metric, the spherical building is not compact unless it is a finite simplicial complex, and hence it is important to give the spherical building a different topology. Besides this natural application to relate two kinds of buildings, this compactification is crucial for the study of cohomology groups of S-arithmetic subgroups in [BoS1]. When the Euclidean building is a tree, this compactification is the well-known compactification of trees by the space of ends.

Spherical buildings with this new topology are called topological buildings by Burns and Spatzier in [BuS2] (actually, the topological buildings were defined slightly differently there) and were used by them to prove rank rigidity of complete manifolds of nonpositive curvature and finite volume in [BuS1]. They were also used by Thorbergsson [Tho2] to classify compact isoparametric submanifolds in \mathbb{R}^n of codimension at least three. One point we want to emphasize in this article is that this *additional structure of a topology* on the spherical Tits buildings is natural and important. In fact, since Tits buildings are combinatorial objects,

their automorphism groups do not have natural topologies and are basically groups with the discrete topology. The additional topology on the buildings allows us to conclude that the automorphism group (or its identity component) is a Lie group, which leads to the desired rigidity results or classifications in differential geometry. For more recent results on uniqueness of topology of the topological buildings, see [GKMW].

Perhaps it is worthwhile to point out that the first major application of the spherical Tits buildings to geometry is in the proof of the Mostow strong rigidity [Mos1]. Though the topological spherical Tits buildings were not introduced or used in [Mos1], the basic idea of topological buildings played an important role there. In fact, the combination of an isomorphism of the two spherical Tits buildings associated with the two Lie groups in question and the homeomorphism on their maximal Furstenberg boundaries was used to prove the isomorphism between the two Lie groups, i.e., the desired Mostow strong rigidity (see the arguments in [Mos1, p. 126]). The point is that the maximal Furstenberg boundaries can naturally be identified with the spaces of the simplexes of maximal dimension (or chambers) of the spherical Tits buildings, and the homeomorphism between them is part of an isomorphism between two topological Tits buildings.

In Thurston's geometrization program of three dimensional manifolds [Thu1] [Thu2] [Thu3], compactness of the space of hyperbolic structures of certain manifolds is crucial. Morgan and Shalen [MS1] [MS2] [Mor2] showed that if they are noncompact, such spaces of hyperbolic structures can be compactified by adding \mathbb{R} -trees admitting certain actions of discrete groups. By analyzing such actions, they could reprove the compactness of the spaces of hyperbolic structures in [Thu1] (see also [Kap]). \mathbb{R} -trees also played an important role in the proof by Otal [Ot2] of a result of Thurston on existence of the hyperbolic metric on certain 3-manifolds fibered over the circle S^1 . The combination of these two results proves the Thurston hyperbolization theorem [Kap] [Ot1] [Ot2]. \mathbb{R} -trees have also been used in other related problems in geometry, topology and combinatorial group theory. In fact, \mathbb{R} -trees were used to understand the automorphism groups of and the topology of the boundaries of Gromov-hyperbolic groups; see [Bes1] [Pau1] [BF] for surveys of these applications together with relations to measured foliations. See also [KIS] [AS] [Wol1] [DDW] for other applications.

The \mathbb{R} -trees also occur naturally in algebraic group theories. In fact, Bruhat and Tits [BT1, §7.4] defined \mathbb{R} -Euclidean buildings which are not simplicial complexes for linear semisimple simply connected algebraic groups over fields with non-discrete valuations. When the rank of the algebraic groups is equal to 1, the \mathbb{R} -Euclidean buildings are \mathbb{R} -trees.

The \mathbb{R} -trees in [MS1] [MS2] can be interpreted as the tangent cone at infinity of the real hyperbolic space. Naturally, the tangent cone at infinity of higher rank symmetric spaces of noncompact type is expected to lead to \mathbb{R} -Euclidean buildings. In fact, they were introduced and used by Kleiner and Leeb in [KL2] to prove the rigidity of irreducible symmetric spaces of rank at least two under quasi-isometries, which is a conjecture of Margulis. For completeness, we mention that the tangent cone at infinity of other rank-1 symmetric spaces of non-compact type is also \mathbb{R} -trees, and the tangent cone at infinity of a (usual) Euclidean building

is also an \mathbb{R} -Euclidean building, and the results in [KL2] also hold for irreducible Euclidean buildings of rank at least two.

Symmetric spaces of noncompact type have played an important role in understanding structures of semisimple Lie groups G and its subgroups, for example, the conjugacy of maximal compact subgroups of G , and the cohomology groups and other problems of discrete subgroups of G . For linear semisimple algebraic groups over non-archimedean local fields, the analogues of the symmetric spaces are the Bruhat-Tits buildings. In fact, the Bruhat-Tits buildings have been used to understand compact open subgroups of semisimple p -adic Lie groups and cohomology groups of S -arithmetic subgroups.

An important application of the Bruhat-Tits buildings in Riemannian geometry is the proof by Gromov and Schoen [GS] of the non-archimedean super-rigidity of co-finite discrete subgroups of $Sp(1, n)$ and $F_{4(-20)}$, the isometry groups of the quaternionic hyperbolic spaces and the Cayley hyperbolic plane. In fact, they developed a theory of harmonic maps from Riemannian manifolds into the Euclidean buildings for this purpose. This result was not covered by the famous Margulis super-rigidity theorem [Mar]. In this application, the fact that Bruhat-Tits buildings are nonpositively curved metric spaces is used crucially. Combined with the archimedean superrigidity for lattices in $Sp(1, n)$ and $F_{4(-20)}$ in [Co], the result of [GS] implies that co-finite discrete subgroups of $Sp(1, n)$ and $F_{4(-20)}$ are arithmetic subgroups.

All the buildings mentioned above appear naturally in geometry and topology. As explained in the beginning of the introduction, an important part of geometry is existence of large symmetry group. It is easy to construct Euclidean buildings of rank 1 whose automorphism group is trivial. For example, any tree without any vertex of valence 1, i.e., with only one edge connected with it, with different valences for different vertices is an Euclidean building with trivial automorphism group. If a thick building has high enough rank, then it will have a large automorphism group. For example, as mentioned before, any thick spherical building of rank at least 3 is the spherical Tits building of a semisimple algebraic group and hence admits a large automorphism group. There is a class of buildings with large symmetry called *Moufang buildings* (or buildings with Moufang property). Briefly, they are buildings with large (root) automorphisms. Spherical buildings of rank at least 3 and Bruhat-Tits buildings over power series fields are Moufang buildings. But there are other Moufang buildings, for example, those associated with infinite dimensional Kac-Moody algebras in [MT]. In fact, they showed that any crystallographic Coxeter group can be realized as the Weyl group of suitable BN -pairs of the automorphism group of some (infinite dimensional) Kac-Moody algebras. Many crystallographic Coxeter groups arise from reflections in the real hyperbolic spaces (see [Vi] [VS] [Ni1] [Ni2] [JS]), in which case the apartment in the building has negative curvature and the building is called a *hyperbolic building* (see [Re3] and its references).

As in the case of Chevalley groups associated with simple complex Lie algebras, there are also Kac-Moody groups associated with Kac-Moody algebras over any field [Ti9] [Re3]. For Kac-Moody groups, a more natural choice of geometry is a twin building, which is a pair of buildings Δ_-, Δ_+ of the same type with a

codistance between every pair of chambers of Δ_-, Δ_+ . The Kac-Moody group acts on $\Delta_- \times \Delta_+$. If the field is finite, then the Kac-Moody group acts properly on $\Delta_- \times \Delta_+$ and can be realized as a cofinite volume discrete subgroup of the locally compact group $Aut(\Delta_-) \times Aut(\Delta_+)$ when the thickness of the buildings is large enough (e.g. if the size of the ground field is greater than the size of the canonical set of generators of the Weyl group). In this case, the buildings Δ_{\pm} are locally finite, and hence the automorphism groups $Aut(\Delta_{\pm})$ have a natural locally compact topology and hence Haar measures. Without the twin buildings, it might be difficult to construct locally compact topological groups that contain the Kac-Moody groups as cofinite discrete subgroups.

We remark (see below) that apartments in spherical buildings have positive curvature, and apartments in Euclidean buildings have zero curvature. Some rigidity results similar to the rigidity of quasi-isometries in [KL2] have been developed for certain hyperbolic buildings in [BP2]. See [BP1] for a survey of related results. On the other hand, hyperbolic buildings have played an increasing important role in combinatorial group theory [CaG1] [CaG2] [Lu] [Re1] [Re2] [Re4] [DJ] [Ja] [Laf1] [Laf2].

For the organization of paper and the topics discussed, see the table of contents at the beginning of this paper. On the other hand, in the rest of the introduction, we outline the main divisions of this paper and the rationales for them.

In §2, we define the spherical Tits buildings and discuss their rigidity properties. We emphasize the classical approach via simplicial complexes and will mention the other more modern approach via W -distance.

In §3, we describe in some detail a striking application of buildings to the Mostow strong rigidity of locally symmetric spaces (or lattices of semisimple Lie groups) when the locally symmetric spaces are irreducible and the symmetric spaces are of rank at least 2 (or lattices are irreducible lattices of semisimple Lie groups of rank at least 2), and an application to the rank rigidity of manifolds of nonpositive curvature (i.e., a characterization of locally symmetric spaces of nonpositive sectional curvature in terms of a notion of rank defined by the dimension of infinitesimal flat subspaces at every point). They make essential use of rigidity properties of the spherical Tits buildings of semisimple Lie groups (or symmetric spaces). For the Mostow strong rigidity, the proof of rank 1 case is very different and does not use Tits buildings. (We note that the spherical Tits building of a symmetric space of rank 1 is a 0-dimensional simplicial complex, i.e., a disjoint union of points, and hence lacks a rich structure).

Though the Margulis super-rigidity is a natural continuation of the Mostow strong rigidity, it involves p -adic Lie groups and hence Bruhat-Tits buildings. Due to this reason, it is put in §5 later. The second part of §3 deals with applications of the spherical Tits buildings to the large scale geometry and compactifications of locally symmetric spaces and cohomology of arithmetic subgroups. In particular, we discuss applications to the Borel-Serre compactification of locally symmetric spaces, to determination of the virtual cohomological dimension of arithmetic subgroups, and to vanishing of simplicial volume of high rank locally symmetric spaces. These applications of spherical Tits buildings have motivated other simplicial complexes in low dimensional topology and geometric group theory such as

the curve complex of surfaces, which will be discussed briefly in §3.11.

In §4, we define Euclidean buildings, the Bruhat-Tits buildings of semisimple p -adic groups, describe some of the basic properties, and discuss their compactifications. One compactification establishes a close connection between Euclidean and spherical buildings as mentioned earlier.

In §5, we discuss applications of Bruhat-Tits buildings to the Margulis super-rigidity and the integral Novikov conjectures of S -arithmetic subgroups. We also mention some applications to harmonic analysis and representation theory of p -adic reductive groups.

After discussing the usual spherical and Euclidean buildings in §2 and §4, we discuss \mathbb{R} -Euclidean buildings, which are non-discrete analogues of usual Euclidean buildings in §6. Since \mathbb{R} -trees are very important special cases of \mathbb{R} -Euclidean buildings, we discuss them and their applications first. Then we recall the general \mathbb{R} -Euclidean buildings and a method in [BT1, §7.4] to construct them from algebraic groups over fields with non-discrete valuations. This method can also be used to construct the usual (simplicial) buildings.

One recent striking application of buildings in geometric group theory is to Kac-Moody groups over finite fields. In §7, we briefly mention twin buildings, Kac-Moody groups over finite fields and some of their properties from the point of view of geometric group theory. We also summarize some results on discrete groups acting on buildings.

Though this paper mainly discusses applications of buildings in geometry and topology, in §8, we indicate briefly other applications in algebraic geometry, potential theory (random walks and Martin compactifications) on buildings, algebraic combinatorics and finite geometry etc together with (hopefully) adequate references.

Before concluding the introduction, we mention that besides the original papers and expository writings of Tits and Bruhat (for example [Ti1] [Ti2]), there are also excellent books [AbB] [Br1] [Br2] [Ro3] [Garr] [TW] [Weis1] [Abr1], survey articles [Ro1] [Ro2] [Sca] [RS], and books consisting of expository articles on buildings and incidence geometries [Bue] [KLPS] [Ros]. Some applications of buildings have been discussed in [AbB] [Br1] [Ro2] [RS]. Some other books on buildings and related topics include [Lan2] [Ma1] [Ma2] [Pa].

On the other hand, many applications of buildings in geometry and topology such as to Kac-Moody groups occurred after the books [Br1] [Ro3] and the paper [RS] were written, and some important applications such as Mostow strong rigidity and the applications of the topological spherical Tits buildings to the rank rigidity of nonpositively curved manifolds and the isoparametric manifolds were mentioned only briefly in [AbB] [RS]. The purpose of this article is to give a more systematical discussion of applications of buildings. We hope to convey to the reader the following points:

1. Buildings are more than simplicial complexes and its global metric properties and topological properties are also important applications.
2. The enhanced topological buildings are more natural and important than the usual combinatorial buildings for applications in differential geometry

and topology.

3. It is instructive and helpful to study and compare types of buildings simultaneously: the (usually locally finite) combinatorial buildings (both spherical and non-spherical), topological buildings, and (non-locally finite) \mathbb{R} -buildings and the particularly important special class of \mathbb{R} -trees. For example, the curve complexes of surfaces were motivated by spherical Tits buildings but share properties with spherical, Euclidean and hyperbolic buildings.
4. Groups become interesting and can be effectively studied by actions on suitable topological spaces, and the algebraic groups and their actions on buildings provide some of most beautiful and fruitful examples of transformation groups.

1.3 Acknowledgments and dedication

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⁶*Buildings and their applications in geometry and topology*, Asian J. Math. 10 (2006), no. 1, 11–80.

Though I never had a chance to talk to the late Professor S.S. Chern, I would like to dedicate this paper discussing many different aspects of geometry as envisioned by Klein and Lie to the memory of my academic grandfather Professor Chern, a great geometer of the twentieth century, on his 100th birthday in 2011.

2 Spherical Tits buildings

In this section, we recall two definitions of spherical Tits buildings, and then construct the spherical Tits building of a semisimple algebraic group over any field using parabolic subgroups, and the general method using BN -pairs or Tits systems. We state the Solomon-Tits Theorem on the homotopy type of buildings, which will be needed for understanding cohomological properties of arithmetic groups later. For applications to rigidity problems, we recall two main results of Tits in [Ti2].

2.1 Definition of buildings as chamber complexes and Solomon-Tits theorem

In this paper, all simplicial complexes are assumed to be finite dimensional.

Definition 2.1.1. A simplex in a simplicial complex is called a chamber if it is maximal, i.e., not contained in another simplex as a face. Two simplexes are called adjacent if they have a common face of codimension 1 (i.e., a facet) and hence of the same dimension. A simplicial complex is called a chamber complex if all chambers have the same dimension, and any two chambers C, C' are connected by a gallery, i.e., a sequence of chambers C_1, \dots, C_n such that $C_1 = C$, $C_n = C'$, and C_i, C_{i+1} are adjacent for $i = 1, \dots, n - 1$.

For example, a tree is a 1-dimensional chamber complex. But two 2-simplexes connected by a 1-simplex in the shape of dumbbell is not.

Definition 2.1.2. A finite group W acting isometrically on a Euclidean space V is called a finite reflection group if it is generated by reflections with respect to hyperplanes.

Clearly, the collection of hyperplanes is finite and invariant under the reflections. We emphasize that the elements in W act as linear transformations. Assume that W acts essentially on V in the sense that there is no nontrivial linear subspace which is fixed by W (see [Br1, Chap I. §1, Chap II. 4, Chap III] for details of definitions and discussions below). Then W is a finite Coxeter group, and any abstract finite Coxeter group can be realized this way.

Recall a Coxeter matrix is a symmetric matrix $M = (m_{ij})$ with $m_{ij} \in \mathbb{N} \cup \{+\infty\}$ satisfying the conditions:

$$m_{ii} = 1, \quad m_{ij} \geq 2 \text{ if } i \neq j.$$

Then the associated Coxeter group W is defined by the following presentation

$$W = \langle r_i \mid (r_i r_j)^{m_{ij}} = 1 \rangle, \quad (2.1.1)$$

where $(r_i r_j)^\infty = 1$, i.e., $m_{ij} = +\infty$, means no relation.

Proposition 2.1.3. *Given any finite Coxeter group W , there is a Coxeter complex, which is a finite chamber complex on which W acts and the action is simply transitive on the set of chambers.*

The Coxeter complex is a finite simplicial complex and can be constructed as follows. Every reflection $\alpha \in W$ fixes a hyperplane \mathcal{H}_α . The collection of such hyperplanes \mathcal{H}_α is invariant under W . Connected components of their complements in V are called chambers, which are simplicial cones. The chambers and their faces together give a partition of V into simplicial cones. Let S be the unit sphere in V . Then the intersection of S with these simplicial cones gives a finite simplicial complex, which is the Coxeter complex of W and whose underlying topological space is S , i.e., a finite triangulation of the unit sphere. It can be seen that a Coxeter complex is a chamber complex.

Definition 2.1.4. A simplicial complex Δ is called a spherical Tits building if it contains a family of subsets called apartments and satisfies the following conditions:

1. Every apartment is a finite Coxeter complex.
2. Any two simplexes are contained in some apartment.
3. Given two apartments Σ and Σ' and simplexes $\sigma, \sigma' \in \Sigma \cap \Sigma'$, there exists an isomorphism of Σ onto Σ' which keeps σ, σ' pointwise fixed.

The condition (3) implies that there is a common Coxeter group whose complex gives the apartments. Since the apartments are triangulations of the unit sphere, the building is called spherical. It also implies that Δ is a chamber complex.

Definition 2.1.5. A spherical Tits building Δ is called *thick* if every simplex of codimension one is contained in at least three chambers.

In the following, all buildings are assumed to be thick unless indicated otherwise. The rank of the building Δ is defined to be $\dim \Delta + 1$.

Example 2.1.6. A simplex of dimension 0 consists of one point, a 0-dimensional Coxeter complex is the disjoint union of two points where the Coxeter group $W \cong \{\pm 1\}$ acts by either fixing both points or swapping them. A spherical Tits building of dimension 0 is a disjoint union of points where every pair of points is an apartment. It is a thick building if and only if it contains at least 4 points. In this case, the building does not carry any more information than the underlying set of points.

By definition, a spherical Tits building of rank r is a union of spheres of dimension $r - 1$. In view of this, the following result is reasonable (see [AbB, Theorem 4.73, p. 197] for references and proofs).

Proposition 2.1.7 (Solomon-Tits Theorem). *As a topological space, a spherical Tits building Δ of rank r has the homotopy type of a bouquet of spheres of dimension $r - 1$, where there is one sphere for every apartment in the building containing a fixed chamber.*

We note that when $r = 1$, a sphere of dimension 0 consists of two points. In the following, we are mainly interested in spherical Tits buildings of dimension at least 1.

2.2 Semisimple Lie groups and buildings

An important example of spherical Tits buildings is constructed from a semisimple algebraic group \mathbf{G} over a field k as follows.

Recall that a subgroup \mathbf{P} of \mathbf{G} defined over k is called a k -parabolic subgroup of \mathbf{G} if \mathbf{G}/\mathbf{P} is a projective variety. For example, when $\mathbf{G} = SL(n)$, the subgroup \mathbf{B} consisting of upper triangular matrices is a k -parabolic subgroup, and more generally, any subgroup of block upper triangular matrices corresponding to a fixed partition of $\{1, \dots, n\}$ is a parabolic subgroup. In fact, any k -subgroup \mathbf{P} containing \mathbf{B} is a k -parabolic subgroup, and any k -parabolic subgroup of \mathbf{G} is conjugate to one which contains \mathbf{B} .

Let $\Delta = \Delta(\mathbf{G})$ be the simplicial complex whose simplexes correspond to proper k -parabolic subgroups as follows:

1. The vertices of Δ correspond to maximal (proper) k -parabolic subgroups of \mathbf{G} .
2. Vertexes $\mathbf{Q}_1, \dots, \mathbf{Q}_m$ form the vertices of a simplex σ if and only if the intersection $\mathbf{Q}_1 \cap \dots \cap \mathbf{Q}_m$ is a k -parabolic subgroup, which corresponds to the simplex σ .

The apartments in the building $\Delta(\mathbf{G})$ are constructed as follows. For any maximal k -split torus \mathbf{T} of \mathbf{G} , there are only finitely many k -parabolic subgroups containing \mathbf{T} , and their corresponding simplexes in Δ form a Coxeter complex whose Coxeter group is given by the Weyl group W of \mathbf{T} , where $W = N(\mathbf{T})/\mathbf{T}$, where $N(\mathbf{T})$ is the normalizer of \mathbf{T} in \mathbf{G} (see [Ti2]). In this building, chambers correspond to minimal parabolic subgroups. Clearly, the group $\mathbf{G}(k)$ of k -points acts on the set of k -parabolic subgroups by conjugation and hence acts on the building $\Delta(\mathbf{G})$ by simplicial automorphisms. The rank of the building $\Delta(\mathbf{G})$ is equal to the k -rank r of \mathbf{G} .

Remark 2.2.1. In this section, we are only concerned with two cases: either $k = \mathbb{R}$ or $k = \mathbb{Q}$. In the former case, instead of the algebraic group \mathbf{G} defined over \mathbb{R} , we could consider only the real locus $G = \mathbf{G}(\mathbb{R})$, a Lie group with finitely many connected components, and the parabolic subgroups of the Lie group G . In the second case, we need the structure of the algebraic group \mathbf{G} defined over \mathbb{Q} in order to study the boundary components for locally symmetric spaces defined by arithmetic subgroups.

Example 2.2.2. When $\mathbf{G} = SL(2)$, every proper parabolic subgroup \mathbf{P} is both maximal and minimal, and hence the simplicial complex $\Delta(\mathbf{G})$ is of dimension 0, i.e., is a disjoint union of points parametrized by the set of proper k -parabolic subgroups of \mathbf{G} . As explained in the previous subsection, it is a spherical building for trivial reasons.

To show that the simplicial complex $\Delta(\mathbf{G})$ defined above satisfies the conditions in Definition 2.1 for spherical buildings, we need the Bruhat decomposition to understand the structure of k -parabolic subgroups. It turns out that they are best described in terms of BN -pairs or Tits systems. In some sense, the conditions for spherical buildings are *the geometric way to describe the group structure on $\mathbf{G}(k)$ imposed by the Bruhat decomposition*.

2.3 BN -pairs or Tits systems, and buildings

The procedure of constructing a building from parabolic subgroups can be formalized as BN -pairs or Tits systems (see [Br1, p. 110]).

Definition 2.3.1. A BN -pair in a group G consists of two subgroups B, N satisfying the following properties:

1. B, N generate G .
2. The subgroup $H = B \cap N$ is normal in N .
3. The quotient group $W = N/H$ is called the Weyl group of the BN -pair and has a generating set S such that
 - (a) for any $s \in S$, $sBs^{-1} \neq B$,
 - (b) for any $s \in S$ and $w \in W$, $BsB BwB \subset BwB \cup BswB$.

We note that the above sets sBs^{-1} , BwB etc. are well-defined even though s, w are elements in the quotient group W . It can be shown that elements in S are of order 2. For any subset $S' \subset S$, let $\langle S' \rangle$ be the subgroup of W generated by S' . Then $B\langle S' \rangle B$ is a subgroup of G (see [Br1, Chap V]).

In the above example of an algebraic group \mathbf{G} defined over k , let \mathbf{N} be the normalizer of \mathbf{T} and \mathbf{B} a minimal k -parabolic subgroup of \mathbf{G} which contains \mathbf{T} . Then $B = \mathbf{B}(k)$ and $N = \mathbf{N}(k)$ form a BN -pair in $\mathbf{G}(k)$.

Given a BN -pair, assume that the Weyl group W is finite. We can construct a spherical building $\Delta = \Delta(G, B)$ as follows. For any subset $S' \subset S$, the corresponding subgroup $B\langle S' \rangle B$ is called a *standard parabolic subgroup* of G with respect to the Tits system (B, N) . A subgroup of G is called a *parabolic subgroup* if it is conjugate to a standard one. The building Δ associated with the BN -pair is the simplicial complex whose simplexes correspond to parabolic subgroups and the incidence relation between the simplexes is the opposite of the inclusion relation of parabolic subgroups. The set of simplexes corresponding to $B\langle S' \rangle B$, where $S' \subset S$, form an apartment Σ of Δ , and the translates $g\Sigma$ under $g \in G$ form a system of apartments. The rank of the building is equal to the number of elements in S .

It can be shown that the conditions for buildings in §2.1 are satisfied by the building $\Delta(B, N)$. It turns out that such buildings constructed from BN -pairs admit more symmetries.

We say that G acts *strongly transitively* on Δ if G acts transitively on the set of pairs (Σ, C) , where Σ is an apartment and C is a chamber in Σ .

Proposition 2.3.2. *Assume that $\Delta(G, B)$ is a thick building. Given a BN -pair, the action of G on $\Delta(G, B)$ is strongly transitive such that B is the stabilizer of a chamber, and N stabilizes an apartment Σ that contains the chamber fixed by B and acts transitively on the set of chambers in Σ , i.e., the Weyl group $W = N/(N \cap B)$ acts simply transitively on the set of chambers in Σ .*

The converse is also true. For a building Δ whose automorphism group acts strongly transitively on it, it can be shown that it is isomorphic to the building associated with a BN -pair in its automorphism group. In fact, fix a labeling of the building Δ (see [Br1, p. 78, Proposition 1], also [Br1, p. 30, p. 72] and §3.3 below). Then the labeling of the vertices gives types to simplexes in Δ . Let G be the automorphism group of Δ that preserves the type of simplexes and the apartment system. Define

$$B = \{g \in G \mid gC = C\}, \quad N = \{g \in G \mid g\Sigma = \Sigma\}.$$

Then

$$H = B \cap N = \{g \in G \mid g \text{ fixes } \Sigma \text{ pointwise.}\}$$

Proposition 2.3.3. *Suppose that G acts strongly transitively on Δ . Then (B, N) is a Tits system in G , and the building Δ is canonically isomorphic to $\Delta(G, B)$.*

For proofs of the above two propositions, see [AbB, Theorem 6.56] and [Br1, Chap V].

Example 2.3.4. When \mathbf{G} is a linear semisimple algebraic group defined over an algebraically closed field k . Let \mathbf{T} be a maximal torus in \mathbf{G} , and \mathbf{B} be a Borel subgroup of \mathbf{G} , i.e., a maximal solvable connected subgroup. Let $B = \mathbf{B}(k)$, and N be the normalizer of $\mathbf{T}(k)$ in $\mathbf{G}(k)$. Then it can be shown that B and N form a BN -pair.

Example 2.3.5. If k is not algebraically closed, then \mathbf{G} may not admit any Borel subgroup defined over k . Let \mathbf{P} be a minimal parabolic subgroup of \mathbf{G} defined over k , and $B = \mathbf{P}(k)$ be the set of its rational points. Let \mathbf{T} be a maximal k -split torus of \mathbf{G} . Let N be the normalizer of $\mathbf{T}(k)$ in $\mathbf{G}(k)$. Then B and N form a BN -pair by the results in [BoT1].

See [AbB, pp. 691-692] [Ti2] for more details and references for the above two examples.

Remark 2.3.6. An important application of BN -pairs is to prove simplicity of finite groups coming from algebraic groups defined over finite fields in [Ti8]. See also [Car1] [Car2] for more detailed expositions. We note that the simplicity criterion in [Ti8] applies to both infinite and finite groups.

2.4 Other definitions of and approaches to buildings

In the previous subsection, we treated buildings as simplicial complexes and chambers complexes, and built them up by starting from vertices. Another important

way to building chamber complexes is to start from chambers and other lower dimension simplices can be constructed from chambers containing them. The key to this combinatorial approach is to understand how chambers are related by elements in the Weyl groups, for example, reflections with respect to their faces of codimension 1.

To make this precise, we use the notion of W -distance. We will follow the definition of [AbB, Definition 5.1.1] (see also [Ro2] [Ro4]) to give the reader a flavor of this approach.

Let W be a Coxeter group with a set of generator S , and let $\ell = \ell_S$ be the length function on W with respect to S , i.e., for any element $w \in W$, $\ell(w)$ is the minimum length of words in S that represent w , i.e.,

$$\ell(w) = \min\{m \mid w = s_1 \cdots s_m, \text{ where } s_1, \dots, s_m \in S\}.$$

Let \mathcal{C} be a nonempty set, whose elements are called chambers. A function $\delta : \mathcal{C} \times \mathcal{C} \rightarrow W$ is called a W -distance function if the following conditions are satisfied:

1. $\delta(C, D) = 1$ if and only if $C = D$.
2. If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w . If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.
3. If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

Definition 2.4.1. A building of type (W, S) is a pair (\mathcal{C}, δ) , where δ is a W -distance on \mathcal{C} .

One important feature of this approach is its close connection with the Bruhat decomposition of groups and the related building [AbB, pp. 308-309].

Definition 2.4.2. Given a group G , a subgroup B , and a Coxeter group W with a set of generators S , then a bijection $C : W \rightarrow B \backslash G / B$ is called a Bruhat decomposition of G with respect to B of the type (W, S) if the following conditions are satisfied:

1. For all $s \in S$ and $w \in W$,

$$C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w).$$

2. If, in addition, $\ell(sw) = \ell(w) + 1$, then $C(s)C(w) = C(sw)$.

Given a Bruhat decomposition of G with respect to B of the type (W, S) , we can construct a building whose set of chambers \mathcal{C} is equal to G/B . In fact, define a W -valued function $\delta : G/B \times G/B \rightarrow W$ by

$$\delta(gB, hB) = C(Bg^{-1}hB) \in W.$$

It can be checked that the conditions for a W -distance function are satisfied.

It is known that the above approach to buildings is equivalent to the earlier one in terms of simplicial complexes and chamber systems [AbB, Chapter 5, §5.6, Theorem 5.91].

The approach to define buildings via W -distance function is also convenient for establishing the Bruhat decomposition. We state [AbB, Theorem 6.34] for the convenience of the reader.

Given a building $\Delta = (\mathcal{C}, \delta)$ defined by a W -distance function δ , an action of a group on \mathcal{C} that preserves the W -distance, i.e., $\delta(gC, gD) = \delta(C, D)$, $C, D \in \mathcal{C}$, is called *Weyl transitive* if for every element $w \in W$, the action of G is transitive on the set of ordered pairs (C, D) of chambers with $\delta(C, D) = w$.

Proposition 2.4.3. *If a group G admits a Weyl-transitive action on a building Δ and B is the stabilizer of a fixed chamber, then G admits a Bruhat decomposition with respect to B , and the building Δ is canonically identified to the building G/B defined above.*

Remark 2.4.4. As mentioned before, given a building (\mathcal{C}, δ) defined by a W -distance function, there is a canonical simplicial complex which is a Tits building and whose chambers are in one-to-one correspondence with elements in \mathcal{C} . This simplicial complex realization of the abstract building (\mathcal{C}, δ) for general infinite Coxeter groups W has some drawbacks for applications in geometric group theory. Different realizations have been given in [Da1], where detailed explanations of the difficulties and constructions are given.

2.5 Rigidity of Tits buildings

An important result of Tits [Ti2, 6.3, 6.13, 8.4.5, 9.1, 10.2] gives a classification of irreducible thick spherical buildings of rank at least 3. For the convenience of the reader, we follow [Sca, §7.3] to state a sample theorem from the classification results.

Recall that a simplicial complex is called a *flag complex* if a subset of vertices forms a simplex if and only if every pair of vertices in that set forms an edge. By assigning a vertex to a linear subspace of a vector space (or equivalently a projective space), and assigning an edge for every pair of subspaces if and only if one is contained in another, we can construct flag complexes.

Theorem 2.5.1. *Every thick spherical Tits building of type \mathbf{A}_n , $n \geq 3$, which means that the Coxeter group W is of type A_n , is isomorphic to the flag complex of a projective space.*

Every spherical Tits building of type \mathbf{D}_n , $n \geq 4$, $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ is the building of the natural BN -pair (or Tits system) of the K -rational points of the split algebraic group (Chevalley) whose Dynkin diagram is of the corresponding type, where K is a unique commutative field.

Every spherical Tits building of type \mathbf{C}_n , $n \geq 4$, is the flag complex of totally isotropic subspaces with respect to a trace valued Hermitian form or (pseudo-)quadratic form.

We note that this is only a sample result on the classification of spherical buildings. For complete statements, see [Ti2] and [Weis2].

The classification of buildings of higher rank is responsible for several rigidity results, for example, characterizations and classification of some finite simple groups of Lie type (see [Su1] [Su2] [Su3] [Har], [Car2]), rank rigidity of manifolds of nonpositive curvature [BuS1] which uses a generalization of Theorem 2.5.1 above (see Theorem 3.3.6 and §3.5 below), and classification of isoparametric submanifolds in \mathbb{R}^n (see [Ter1] [Tho2] and §3.7). Though the applications are different, the strategy is the same and consists of two steps:

1. Construct a building or BN -pair from the given groups or manifolds.
2. Apply the classification result to determine the building and use it to recover the original object.

Another important result of Tits shows that group structures can be fully recovered from the buildings through description of morphisms between them [Ti2, Theorem 5.8] [Mos, Theorem 16.1, Corollary 16.2].

Theorem 2.5.2. *Let \mathbf{G}, \mathbf{G}' be linear semisimple groups defined over \mathbb{R} which have no center and no simple factors of \mathbb{R} -rank less than or equal to 1. For any isomorphism between the spherical Tits buildings $\psi : \Delta(\mathbf{G}) \rightarrow \Delta(\mathbf{G}')$, then there exists an isomorphism between $\mathbf{G}(\mathbb{R})$ and $\mathbf{G}'(\mathbb{R})$ as abstract groups that induces the map ψ on the buildings.*

This shows that the building $\Delta(\mathbf{G})$ determines the group $G = \mathbf{G}(\mathbb{R})$. Since parabolic subgroups of G describe the structures of G at infinity or large scale geometry of G , this also reflects some rigidity property of G . In fact, this is used crucially in the proof of the Mostow strong rigidity in [Mos1] (see Theorem 3.4.1).

Remark 2.5.3. It should be emphasized that the building $\Delta(\mathbf{G})$ does not obviously determine G as a Lie group. In fact, by Theorem 2.5.2, $\mathbf{G}(\mathbb{R})$ is isomorphic to $\mathbf{G}'(\mathbb{R})$ as abstract groups, $\varphi : \mathbf{G}(\mathbb{R}) \rightarrow \mathbf{G}'(\mathbb{R})$. If \mathbf{G} is almost simple over \mathbb{R} , by [BoT2], then φ is essentially given by an isogeny $\mathbf{G} \rightarrow \mathbf{G}'$, since any automorphism of \mathbb{R} is trivial (It follows from the fact that any automorphism preserves well-orderedness and rational numbers). In this case, the topology of $G = \mathbf{G}(\mathbb{R})$ is determined by the building $\Delta(\mathbf{G})$. On the other hand, if G is a *complex* Lie group, then the topology of G is not determined by $\Delta(\mathbf{G})$ due to nontrivial automorphisms of \mathbb{C} over \mathbb{R} . See [Kr5] for more discussions and more general results on the uniqueness of topology of a simple Lie group.

If we enrich the spherical Tits buildings to topological spherical Tits buildings as in [BuS2] (see Definition 3.3.1 below), then we can recover the topologies of G and G' and should be able to show that they are isomorphic as Lie groups. This was used in [Mos1, p. 126] in the proof of Mostow strong rigidity. This is the reason that we emphasize topological buildings in this paper. As mentioned in the introduction, the idea of topological buildings was used implicitly in the proof of the Mostow strong rigidity since the homeomorphism on the maximal Furstenberg boundaries is crucial to the proof.

3 Geometric realizations and applications of spherical Tits buildings

In this section, we discuss a geometric realization of the spherical Tits building of semisimple Lie groups, define the related topological Tits buildings, and describe applications of buildings to rigidity problems in differential geometry, and global geometry and compactifications of symmetric and locally symmetric spaces.

More specifically, for real semisimple Lie groups, we realize their spherical buildings in terms of the sphere at infinity of the corresponding symmetric space of noncompact type. This naturally leads to the concept of topological Tits buildings. Then we recall the maximal Satake compactification of symmetric spaces in order to introduce the maximal Furstenberg boundary, which is used crucially in the proof of the Mostow strong rigidity, and explain relations between the boundaries of compactifications of symmetric spaces and the Tits buildings in [GJT] [BJ1].

Then we outline the proof of the Mostow strong rigidity in the higher rank case, which will motivate the Margulis super-rigidity in §5 and the Margulis conjecture on quasi-isometries between symmetric spaces and quasi-flats in §6.

To illustrate the power of topological Tits buildings, we discuss their applications to the problem of rank rigidity of manifolds of nonpositive curvature and finite volume, and the classification of compact isoparametric submanifolds of codimension at least 3 in Euclidean spaces. These are major applications of the (topological) Tits buildings of real Lie groups.

Then we discuss the spherical Tits buildings of semisimple linear algebraic groups defined over \mathbb{Q} . Quotients of the Tits buildings by arithmetic subgroups occur naturally as the boundary of a compactification of the arithmetic locally symmetric spaces, called the Tits compactification, which is homeomorphic to the Gromov compactification and the geodesic compactification of locally symmetric spaces, and is also related to the tangent cone at infinity of the locally symmetric spaces and the Siegel conjecture on comparison of metrics on Siegel sets.

Finally we describe how the Borel-Serre compactification of locally symmetric spaces and the Solomon-Tits theorem of buildings can be used to determine the duality property and the virtual cohomological dimension of arithmetic subgroups of semi-simple Lie groups. We also explain how these applications motivated curve complexes of surfaces which have played a fundamental role in the low dimensional topology and mapping class groups. This is another indication of that simplicial complexes arise naturally in describing geometry and topology at infinity of important noncompact spaces such as Teichmüller spaces of Riemann surfaces.

3.1 Geodesic compactification of symmetric spaces

Let G be a linear semisimple Lie group, $K \subset G$ a maximal compact subgroup, and $X = G/K$ the associated Riemannian symmetric space. For simplicity, we endow X with the invariant metric induced from the Killing form. In this subsection, we will realize the spherical Tits building $\Delta(G)$ using the asymptotic geometry of X and show how it can in turn be used to understand the geometry at infinity of X and compactifications of X .

It is known that X is a Riemannian symmetric space of noncompact type and hence is simply connected and of nonpositive sectional curvature, i.e., a Hadamard manifold. It admits the geodesic compactification $X \cup X(\infty)$ which is defined as follows.

All geodesics considered in this paper are of unit speed and directed. Two geodesics $\gamma_1(t), \gamma_2(t)$, $t \in \mathbb{R}$, in X are called *equivalent* if

$$\limsup_{t \rightarrow +\infty} d(\gamma_1(t), \gamma_2(t)) < +\infty.$$

Denote the set of equivalence classes of geodesics by $X(\infty)$.

Let $x_0 \in X$ be a basepoint in X . Then it can be shown that in each equivalence class, there is a unique geodesic passing through x_0 , and hence $X(\infty)$ can be identified with the unit sphere in the tangent space $T_{x_0}X$. Due to this identification, $X(\infty)$ is often called the *sphere at infinity*.

The topology of the compactification $X \cup X(\infty)$ is defined such that an unbounded sequence y_j in X converges to an equivalence class $\xi \in X(\infty)$ if the sequence of geodesics γ_j passing through x_0 and y_j converges to a geodesic γ_∞ which belongs to ξ . It can be checked easily that $X \cup X(\infty)$ is a compact Hausdorff space and is called the *geodesic compactification* of X .

Example 3.1.1. If $X = SU(1, 1)/U(1)$ is the Poincaré disk, $D = \{z \in \mathbb{C} \mid |z| < 1\}$ with the metric $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$, then the set of equivalence classes of geodesics can be canonically identified with the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and the geodesic compactification $X \cup X(\infty)$ is equal to the closed unit disk \bar{D} . It is well-known that the Möbius transformations (or fractional linear transformations) that preserve the unit disk D extends to the closed disk \bar{D} .

If we identify X with the open unit ball in $T_{x_0}X$ by the exponential map and shrinking along rays from the origin, then $X \cup X(\infty)$ is homeomorphic to the closed unit ball in $T_{x_0}X$. But the intrinsic definition of the compactification $X \cup X(\infty)$ allows us to see that any isometry on X extends continuously to $X \cup X(\infty)$. Hence we have

Proposition 3.1.2. *The G -action on X extends continuously to $X \cup X(\infty)$, and the geodesic compactification does not depend on the choice of base point x_0 .*

This action allows one to understand parabolic subgroups of G geometrically.

Proposition 3.1.3. *For any point $\xi \in X(\infty)$, its stabilizer $G_\xi = \{g \in G \mid g\xi = \xi\}$ is a parabolic subgroup. Conversely, any (proper) parabolic subgroup P fixes some boundary point in $X(\infty)$.*

For proofs, see [GJT] [BJ1] [BGS]. For each (proper) parabolic subgroup P , let σ_P be the set of points in $X(\infty)$ whose stabilizer in G is exactly equal to P . It can be shown that the set of points in $X(\infty)$ fixed by P is exactly equal to the closure of σ_P . Unless P is a maximal proper parabolic subgroup, σ_P is not closed.

Proposition 3.1.4. *For any two parabolic subgroups P_1, P_2 , $\sigma_{P_1}^o$ and $\sigma_{P_2}^o$ are either disjoint or identical. Furthermore, σ_{P_1} is contained in the closure of σ_{P_2} in*

$X(\infty)$ if and only if P_1 contains P_2 ; and σ_P consists of a point if and only if P is a maximal parabolic subgroup; and each σ_P has a natural (spherical) simplicial structure under the identification of $X(\infty)$ with the unit sphere in $T_{x_0}(X)$.

See [BJ1, §I.2] for a proof. An immediately corollary of the above proposition is

Proposition 3.1.5. *The disjoint decomposition $X(\infty) = \cup_P \sigma_P$ gives a simplicial complex isomorphic to the spherical Tits building $\Delta(G)$ of G .*

It should be emphasized that the simplicial topology on $\Delta(G)$ is not the same as the topology of $X(\infty)$ when it is identified with the unit sphere in $T_{x_0}X$. For example, when X is a symmetric space of rank 1, $X(\infty)$ is a closed (real analytic) manifold diffeomorphic to the unit sphere in $T_{x_0}X$. On the other hand, in this case, every proper parabolic subgroup of G is both a maximal and minimal parabolic subgroup, and $\Delta(G)$ is the disjoint union of points in $X(\infty)$, i.e., $X(\infty)$ with the discrete topology, and hence $\Delta(G)$ induces the discrete topology on $X(\infty)$.

3.2 Buildings and compactifications of symmetric spaces

This identification of $\Delta(G)$ with the boundary $X(\infty)$ in Proposition 3.1.5 is important for several reasons:

1. Since $X(\infty)$ is defined in terms of asymptotic classes of geodesics, the building $\Delta(G)$ describes the asymptotic geometry at infinity of X .
2. The boundary $X(\infty)$ of the compactification $X \cup X(\infty)$ is assembled from the boundary pieces σ_P where P runs over parabolic subgroups of G .

The conclusion in (2) suggests that if we change the boundary component σ_P of the parabolic subgroup P , we could get other compactifications \overline{X} of X . This is indeed the case, and hence the boundaries $\partial\overline{X}$ of the compactifications \overline{X} often have a cell-complex structure related to the spherical Tits building $\Delta(G)$, or rather the boundaries are unions of boundary components parametrized by the Tits building.

To illustrate this, we discuss the maximal Satake compactification \overline{X}_{\max}^S of X to show that its boundary has a cell-complex structure dual to the spherical Tits building $\Delta(G)$ (for more details and proofs of the following discussions, see [GJT] and [BJ1]).

Let

$$\tau : G \rightarrow SL(n, \mathbb{C})$$

be a finite dimensional irreducible and faithful representation satisfying the condition:

$$\tau(\theta(g)) = (\tau(g)^*)^{-1}, \quad g \in G,$$

where θ is the Cartan involution of G associated with the maximal compact subgroup K . Then τ induces a G -equivariant embedding

$$i_\tau : X = G/K \rightarrow SL(n, \mathbb{C})/SU(n), \quad gK \mapsto \tau(g)\tau(g)^*.$$

The quotient $SL(n, \mathbb{C})/SU(n)$ is the symmetric space of positive definite Hermitian matrices of determinant 1 and is hence contained in the real vector space \mathcal{H}_n of Hermitian $n \times n$ -matrices. Let $P(\mathcal{H}_n)$ be the associated real projective space. By composing with i_τ , we obtain an embedding

$$i_\tau : X \rightarrow P(\mathcal{H}_n),$$

and the closure of $i_\tau(X)$ in $P(\mathcal{H}_n)$ is called the Satake compactification of X associated with the representation ρ , denoted by \overline{X}_τ^S . Since G acts on \mathcal{H}_n through the representation ρ and the embedding i_τ is G -equivariant, the G -action on X extends to a continuous action on \overline{X}_τ^S .

As a topological G -space, \overline{X}_τ^S only depends on the relative position of the highest weight μ_τ of τ , i.e., on the Weyl chamber face which contains μ_τ as an interior point. When μ_τ is generic, i.e., contained in the interior of the Weyl chamber, the compactification \overline{X}_τ^S is called the *maximal Satake compactification* and denoted by \overline{X}_{\max}^S .

The boundary $\partial\overline{X}_{\max}^S$ can be decomposed into boundary components parametrized by parabolic subgroups. For each parabolic subgroup P of G , let N_P be the unipotent radical of P , A_P the split component invariant under the Cartan involution θ associated with K , and

$$P = N_P A_P M_P \cong N_P \times A_P \times M_P \quad (3.2.1)$$

be the Langlands decomposition of P . Note that the Langlands decomposition is a diffeomorphism but not a group isomorphism. For example, when $G = SL(n, \mathbb{R})$, and

$$P = \left\{ g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in M_k(\mathbb{R}), C \in M_{n-k}(\mathbb{R}), B \in M_{k \times n-k}(\mathbb{R}), g \in SL(n, \mathbb{R}) \right\},$$

then

$$\begin{aligned} N_P &= \left\{ \begin{pmatrix} I_k & B \\ 0 & I_{n-k} \end{pmatrix} \mid B \in M_{k \times n-k} \right\}, \\ A_P &= \left\{ \begin{pmatrix} aI_k & 0 \\ 0 & cI_{n-k} \end{pmatrix} \mid a, c \in \mathbb{R}^\times, a^k c^{n-k} = 1 \right\}, \\ M_P &= \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \mid \det A \det C = 1 \right\}. \end{aligned}$$

Define

$$X_P = M_P / K \cap M_P, \quad (3.2.2)$$

which is a lower dimensional symmetric space of noncompact type, called the boundary symmetric space associated with P . In the above example of P in $SL(n, \mathbb{R})$, the boundary symmetric space is

$$X_P = (SL(k, \mathbb{R})/SO(k)) \times (SL(n-k, \mathbb{R})/SO(n-k)).$$

Then as a set,

$$\overline{X}_{\max}^S = X \cup \coprod_P X_P.$$

For any two parabolic subgroups P_1, P_2 , the symmetric space X_{P_1} is contained in the closure of X_{P_2} if and only if P_1 is contained in P_2 . Since each X_P is a cell, the boundary $\partial\overline{X}_{\max}^S$ is a cell-complex dual to the spherical Tits building $\Delta(G)$ and the topology of the boundary components X_P is determined by the topology of the topological building of G .

Remark 3.2.1. There are also other compactifications of X , for example the Martin compactification, whose boundaries have cell-complex structure of more complicated type. The relations between the Tits buildings and structures of the boundary components of compactifications already exist on closures of maximal flat totally geodesic submanifolds, called flats, in symmetric spaces. For example, the boundary of a flat in the maximal Satake compactification \overline{X}_{\max}^S is a polyhedral compactification whose boundary is dual to an apartment in the spherical Tits building. See [GJT] and [BJ1] for more details.

In \overline{X}_{\max}^S , there are only finitely many G -orbits corresponding to the finitely many conjugacy classes of parabolic subgroups. In fact, each boundary component is contained in a G -orbit as a proper subset and the boundary symmetric spaces of conjugate parabolic subgroups belong to the same G -orbit. The orbit for the minimal parabolic subgroups is the only closed one and is equal to G/P_0 , where P_0 is a minimal parabolic subgroup.

Definition 3.2.2. Let P_0 be a minimal parabolic subgroup of G , the homogeneous space G/P_0 is called the maximal Furstenberg boundary of X or G , and denoted by \mathcal{X}_0 .

Remark 3.2.3. The fact that the maximal Furstenberg is contained canonically in \overline{X}_{\max}^S is important in the proof of Mostow rigidity to be discussed below. It should be pointed out that when the rank of X is greater than or equal to 2, the sphere at infinity $X(\infty)$ (or the boundary of the geodesic compactification) contains infinitely many G -orbits which are isomorphic to the maximal Furstenberg boundary. The maximal Furstenberg boundary \mathcal{X}_0 is closely related to the Tits building. In fact, the set of chambers in the Tits building $\Delta(G)$ is parametrized by \mathcal{X}_0 . Its canonical embedding in \overline{X}_{\max}^S explains why the maximal Satake compactification rather than the geodesic compactification $X \cup X(\infty)$ is used in the Mostow strong rigidity below.

3.3 Topological spherical Tits buildings and Moufang buildings

As explained in the previous subsection, the boundary of a compactification of the symmetric space X , for example, the geodesic compactification $X \cup X(\infty)$ and the maximal Satake compactification \overline{X}_{\max}^S , has the structure of cell-complex closely related to the spherical Tits building of X .

On the other hand, the boundary of the compactification is a compact topological space. When $X = SL(2, \mathbb{R})/SO(2)$ is the Poincaré upper half plane, or equivalently the unit disc, the Tits building $\Delta(G)$ is the unit circle with the discrete topology. On the other hand, the boundary $X(\infty)$ is the unit circle with the usual topology.

The unit circle $X(\infty)$ with the usual topology is the topological spherical Tits building corresponding to the building $\Delta(G)$.

In fact, topological buildings are defined in [BuS2, Definition 1.1] as follows. Let Δ be a spherical Tits building of rank r . Then any chamber of Δ has r vertices. Fix a chamber C and list its vertices as v_1, \dots, v_r . Then any other chamber has also a unique induced ordering of its vertices. Similarly, the vertices of every simplex σ in Δ have also a well-defined ordering. For each $n = 1, \dots, r$, let Δ_n be the set of simplexes with n vertices. Then Δ_1 is the set of vertices, and Δ_r is the set of chambers. The above ordering of vertices gives a well-defined map

$$\Delta_n \rightarrow (\Delta_1)^n, \quad \sigma \mapsto (x_1, \dots, x_n), \quad (3.3.1)$$

where x_1, \dots, x_n are the vertices of σ listed according to the order.

Definition 3.3.1. A topological spherical Tits building is a spherical Tits building Δ with a Hausdorff topology on Δ_1 such that the image of Δ_n in $(\Delta_1)^n$ under the map in Equation (3.3.1) is a closed subset.

In a topological building, we endow Δ_n with the subset topology induced from $(\Delta_1)^n$. We can also define a topological Tits building as a Tits building with a topology on every Δ_n such that the map in Equation (3.3.1) is an embedding with a closed image.

If the topology of a topological building is metrizable, it is called a *metric building* [BuS2, p. 12]. Many naturally occurring spherical Tits buildings such as $\Delta(G)$ are metric buildings.

Identify $X(\infty)$ with the unit sphere in $T_{x_0}X$ and denote the subset distance induced from the Riemannian norm on $T_{x_0}X$ by d . Then d induces the Hausdorff distance on the collection of subsets of $X(\infty)$ by:

$$d_H(A, B) = \inf\{\delta \mid A \subseteq N_\delta(B), B \subseteq N_\delta(A)\},$$

where $N_\delta(B)$ is the δ -neighborhood of B .

Realize $\Delta(G)$ as a simplicial complex on $X(\infty)$. Then the Hausdorff distance function d_H defines a metric on $\Delta_1, \dots, \Delta_r$, where r is the rank of G or $\Delta(G)$. It can be checked easily that with this metric on Δ_1 , $\Delta(G)$ is a metric spherical Tits building and hence is a topological spherical Tits building.

Definition 3.3.2. A topological spherical Tits building Δ of rank r is called *compact, connected, locally connected* if the topological space Δ_r has the corresponding property. An automorphism of a topological Tits building is an automorphism of the building whose restriction to each Δ_n , $n = 1, \dots, r$, is a homeomorphism.

Remark 3.3.3. On a topological building Δ , we can put a topology on the whole underlying space of the building Δ by gluing up the topologies on the subspaces

Δ_n , $n = 1, \dots, r$. Then an automorphism is a homeomorphism that preserves the simplicial structure. Other notions such as compactness can also be defined directly in terms of Δ . This might be more useful for the compactification of the Bruhat-Tits buildings by the spherical Tits buildings in [BoS1] (see also §3.5 below), since the spherical Tits buildings are topological in the sense here.

Two important results on topological buildings in [BuS2] are given in the next two theorems.

Theorem 3.3.4. *If Δ is an irreducible compact metric building of rank at least 2, then its topological automorphism group is locally compact in the compact open topology.*

Before stating the next result, we need the notion of *Moufang buildings*. For any apartment $\Sigma \subset \Delta$, a root hyperplane in Σ divides the apartment into two half planes, which are often called roots.⁷ The *star* of a simplex $A \in \Delta$ is the set of simplices joinable to A and is denoted by $\text{st}_\Delta A$. For a root α of a building Δ , let U_α be the group consisting of all elements of $g \in G = \text{Aut}(\Delta)$ satisfying the conditions:

1. g fixes every point, in particular all the chambers, in α .
2. g fixes $\text{st}_\Delta P$ pointwise for every panel $P \in \alpha - \partial\alpha$.

The group U_α is called the *root group associated with the root α* [AbB, Definition 7.24].

Given a root α and a panel $P \subset \partial\alpha$, the set of chambers of a building Δ that has P as a face is denoted by \mathcal{C}_P . Let C be the unique chamber in α that has P as a face, and define $\mathcal{C}(P, \alpha) = \mathcal{C}_P - \{C\}$. See [AbB, Definition 7.1].

Definition 3.3.5. In the above notation, a building Δ is called a *Moufang building* if for every root α , U_α acts transitively on the set of all apartments containing α , which is equivalent to that for every panel $P \subset \alpha$, U_α also acts transitively on $\mathcal{C}(P, \alpha)$.

In particular, a Moufang building Δ has abundant automorphisms. This Moufang condition is the one given in [AbB, Definition 7.27] [Ti2], [Ro2] and [Weis1, Definition 11.2]. The definition in [BuS2, p.22] is slightly different by imposing a weaker condition in the definition of root groups.

Theorem 3.3.6. *If Δ is an infinite, irreducible, locally connected, compact, metric Moufang spherical building of rank at least 2, then the identity component G^0 of the topological automorphism group of Δ is a simple noncompact real Lie group without center, and Δ is isomorphic to the topological spherical Tits building $\Delta(G^0)$.*

⁷Abstractly, a chamber complex is called thin if every panel (i.e., a simplex of codimension 1) is a face of exactly two chambers. A folding of a thin complex Σ is an idempotent endomorphism ϕ , i.e., $\phi^2 = \phi$, such that for every chamber $C \in \phi(\Sigma)$, there is exactly one chamber $C' \in \Sigma - \phi(\Sigma)$ with $\phi(C') = C$. A root of Σ is a subcomplex α that is the image of a reversible folding. (A reversible folding means a folding ϕ such that the “folding on chambers” opposite to ϕ can be extended to a folding of the thin complex. See [AbB, Definition 3.48] for details.)

This theorem [BuS2] is similar to Theorem 2.5.1. In fact, in Theorem 2.5.1, the building is required to be of rank at least three, which together with the other conditions there implies that the building is Moufang. Hence the conditions in Theorem 3.3.6 of rank being at least 2 and of being Moufang is weaker. This weakening to rank at least 2 is crucial for application to the rank rigidity of manifolds of nonpositive curvature in [BuS1] (see §3.5 below).

3.4 Mostow strong rigidity

The fundamental result of Mostow rigidity [Mos1] concerns the rigidity of finite volume locally symmetric spaces of higher rank. It says roughly that the fundamental group determines the space isometrically up to scaling factors. This is a major application of the result of Tits on rigidity (or classification) of spherical Tits buildings of higher rank.

Let G be a connected semisimple Lie group without center and with no nontrivial compact factor, $K \subset G$ a maximal compact subgroup, and $X = G/K$ the associated symmetric space as before. Any torsion free discrete subgroup Γ of G acts freely and properly on X , and the quotient $\Gamma \backslash X$ is a smooth manifold, called a locally symmetric space.

The discrete subgroup Γ is called a *lattice* if $\Gamma \backslash X$ has finite volume and *uniform* if $\Gamma \backslash X$ is compact. It is called *reducible* if G admits a nontrivial decomposition $G = G_1 \times G_2$, which implies an isometric decomposition $X = X_1 \times X_2$, and two subgroups $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ such that Γ is commensurable to $\Gamma_1 \times \Gamma_2$, i.e., $\Gamma \cap \Gamma_1 \times \Gamma_2$ has finite index in both Γ and $\Gamma_1 \times \Gamma_2$. In this case, after lifting to a finite cover, $\Gamma \backslash X$ splits as a product of two locally symmetric spaces. An equivalent definition is that the images of Γ in G_1 , G_2 are discrete [Mos, p. 133]. If Γ is not reducible, it is called *irreducible*.

We recall that a *flat* in X is a totally geodesic flat subspace in X . It is known that G acts transitively on the set \mathcal{F} of maximal flats X , and hence the maximal flats in X have the same dimension, which is equal to the rank of G . For convenience, a flat means a maximal flat in X in the following unless indicated otherwise.

The precise statement of the Mostow strong rigidity [Mos1] is given in the following theorem.

Theorem 3.4.1. *Let $X = G/K$ be a symmetric space of noncompact type as above, and Γ an irreducible uniform lattice in G . Assume that either the rank of X is at least 2, or the rank of X is equal to 1 and $\dim X \geq 3$. Suppose that $X' = G'/K'$ is a symmetric space of noncompact type and Γ' is a uniform lattice acting on X' . If Γ' is isomorphic to Γ (as abstract groups), then $\Gamma \backslash X$ and $\Gamma' \backslash X'$ are isometric up to suitable normalizing scalars of the irreducible factors of X , which implies that G and G' are isomorphic as Lie groups.*

This theorem says that under the conditions in the theorem, the fundamental group of $\Gamma \backslash X$ determines $\Gamma \backslash X$ isometrically up to suitable scaling. It should be pointed out that both conditions are necessary. For example, if $G = SL(2, \mathbb{R})$, $X = SL(2, \mathbb{R})/SO(2)$, the upper half plane, the rank of X is equal to 1, and the

strong rigidity of Γ fails due to the high dimension of the moduli spaces of Riemann surfaces (or the deformation space of $\Gamma \backslash X$). Now take $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, the associated symmetric space X has rank equal to 2. Take a reducible lattice $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1, \Gamma_2 \subset SL(2, \mathbb{R})$ are uniform lattices. Due to the failure of the strong rigidity for Γ_i , it also fails for such reducible Γ .

Remark 3.4.2. For a history of the results leading to the Mostow strong rigidity, see [Mos, p. 5]. A generalization of this theorem is given in [BGS]. In fact, when the rank of X is at least 2, the target manifold $\Gamma' \backslash X'$ can be replaced by any compact manifold with nonpositive sectional curvature. The assumption on the rank is necessary due to existence of exotic negatively curved metrics on locally symmetric spaces of rank 1 (see [On] [FJO] [AF]). See [Sp1] [Sp2] for surveys of other aspects of rigidity results, in particular, from the point of view of dynamic systems, which is a huge area.

In the case of rank at least 2, Theorem 3.4.1 is proved using the result in Theorem 2.5.2 on Tits buildings. In the case of rank 1, the proof is different and uses generalizations of quasi-conformal maps on the sphere at infinity $X(\infty)$. See Remark 3.4.10 for some brief comments and references.

Since the rank 1 case does not make use of buildings, we will only briefly outline the proof for the case of higher rank. The proof in [Mos1] proceeds in the following steps:

1. Since X, X' are contractible, $\Gamma \backslash X$ and $\Gamma' \backslash X'$ are $K(\Gamma, 1)$ -spaces. The isomorphism between Γ and Γ' induces a homotopy equivalence

$$\psi : \Gamma \backslash X \rightarrow \Gamma' \backslash X', \quad (3.4.1)$$

which induces a Γ -equivariant quasi-isometry

$$\varphi : X \rightarrow X'. \quad (3.4.2)$$

2. Let $\mathcal{X}_0, \mathcal{X}'_0$ be the maximal Furstenberg boundaries sitting in the boundary of the maximal Satake compactifications \overline{X}_{\max}^S and \overline{X}'_{\max}^S . The quasi-isometry φ induces a homeomorphism

$$\varphi_0 : \mathcal{X}_0 \rightarrow \mathcal{X}'_0,$$

which is called the boundary value map of φ .

3. The same proof as in (2) shows that φ induces an isomorphism between the spherical Tits buildings $\Delta(G)$ and $\Delta(G')$.
4. Apply Theorem 2.5.2 to conclude $G \cong G'$ as abstract groups. Combining with (2), it implies that G and G' are isomorphic as Lie groups.

Recall that a map $\varphi : X \rightarrow X'$ is called a quasi-isometry if there exist positive constants L, C such that for all $x_1, x_2 \in X$,

$$L^{-1}d(x_1, x_2) - C \leq d'(\varphi(x_1), \varphi(x_2)) \leq Ld(x_1, x_2) + C, \quad (3.4.3)$$

where d, d' are the distance functions of X and X' respectively, and for every $x' \in X'$,

$$d'(x', \varphi(X)) \leq C. \quad (3.4.4)$$

Then Step (1) basically follows from the assumption that $\Gamma \backslash X$ and $\Gamma' \backslash X'$ are compact and X, X' are contractible.

The proof of Step (2) depends on the identification of the maximal Furstenberg boundary \mathcal{X}_0 with the set of equivalence classes of Weyl chambers in flats \mathcal{F} in X .

Specifically, for any basepoint $x_0 \in X$, any flat F in X passing through x_0 is of the form $e^{\mathfrak{a}}x_0$, where \mathfrak{a} is a maximal abelian subalgebra in \mathfrak{p}_{x_0} , where \mathfrak{p}_{x_0} is the component invariant under θ_0 in the Cartan decomposition $\mathfrak{g} = \mathfrak{k}_{x_0} + \mathfrak{p}_{x_0}$, where θ_0 is the Cartan involution θ_0 associated with x_0 . The connected components of the complement of the root hyperplanes in \mathfrak{a} are called Weyl chambers. For any chamber C of \mathfrak{a} , the image $e^C x_0$ in X is called a chamber in X , which is also denoted, for simplicity, by C . Two chambers C_1, C_2 are called equivalent (or asymptotic) if

$$d_H(C_1, C_2) < +\infty, \quad (3.4.5)$$

where d_H is the Hausdorff distance on subsets of X . For each chamber C , denote its equivalence class by $[C]$. Then the set of equivalence classes of Weyl chambers $[C]$ can naturally be identified with the maximal Furstenberg boundary \mathcal{X}_0 [Mos, Lemma 4.1]. The identification comes from the following facts:

1. For any chamber C in X and a sequence x_j in C which diverges to infinity and its distance to the chamber walls also goes to infinity, then x_j converges to a point $\xi \in \mathcal{X}_0$. Hence, there is a unique point in \mathcal{X}_0 corresponding to each chamber C .
2. If two chambers are equivalent, then they correspond to the same point.

To prove the existence of the boundary value map φ_0 , it suffices to show that for any chamber C in X , there is a chamber C' in X and hence a unique equivalence class $[C']$ of chambers such that

$$d_H(C', \varphi(C)) < +\infty. \quad (3.4.6)$$

Then we define the map φ_0 by

$$\varphi_0([C]) = [C'].$$

For this purpose, a crucial role is played by behaviors of flats under quasi-isometries. First we note that the rank of X can be characterized algebraically in terms of Γ [Mos, Lemma 11.3].

Proposition 3.4.3. *Under the assumption of Theorem 3.4.1, the rank of G and hence of X is equal to the maximal rank of abelian subgroups of Γ . Hence, the rank of X is equal to the rank of X' .*

Let \mathcal{F}' be the set of flats in X' .

Proposition 3.4.4. *Let $\varphi : X \rightarrow X'$ be the Γ, Γ' -equivariant quasi-isometry constructed in Equation 3.4.2. Then for any flat $F \in \mathcal{F}$, there is a unique flat $F' \in \mathcal{F}'$ such that*

$$d_H(F', \varphi(F)) < +\infty.$$

Define the map

$$\bar{\varphi} : \mathcal{F} \rightarrow \mathcal{F}', \quad F \mapsto F'.$$

Then $\bar{\varphi}$ is a homeomorphism.

This is proved in [Mos, Lemma 14.1]. To apply this proposition to prove the chamber approximation in Equation 3.4.6, we need to characterize chambers in X in terms of asymptotic intersection of flats in the geodesic compactification $X \cup X(\infty)$. In fact, Mostow introduced *splices* in [Mos, p. 107] in terms of asymptotic intersection of flats $F_0 \cap_x F$ [Mos, p. 56], where $x \in F$ and

$$F_0 \cap_x F = \cup \{ \text{rays in } F \text{ starting from } x_0 \text{ and contained in a bounded neighborhood of } F_0 \}.$$

It can be seen that Weyl chambers and their faces are splices. Then he introduced the notion of *irreducible splices* and showed that they are given by chambers and chamber faces.

Remark 3.4.5. By definition, $F_0 \cap_x F$ clearly depends on the choice of the base-point x . On the other hand, its boundary points in the geodesic compactification $X \cup X(\infty)$, i.e., the intersection

$$\overline{F_0 \cap_x F} \cap X(\infty)$$

is independent of x , where $\overline{F_0 \cap_x F}$ is the closure of $F_0 \cap_x F$ in $X \cup X(\infty)$. Furthermore, this intersection with $X(\infty)$ can be shown to be equal to the intersection

$$\overline{F_0} \cap \overline{F} \cap X(\infty),$$

and determines $F_0 \cap_x F$. For this reason, we call $F_0 \cap_x F$ the asymptotic intersection of the flats F_0 and F .

Combining Proposition 3.4.4 with the observation that Weyl chambers and their faces are given by irreducible splices, Mostow proved [Mos, Theorem 15.2] the following result.

Proposition 3.4.6. *Given any chamber C in X , there exists a chamber C' in X' such that*

$$d_H(C', \varphi(C)) < +\infty,$$

and hence there is a well-defined map

$$\varphi_0 : \mathcal{F} \rightarrow \mathcal{F}'_0, \quad [C] \rightarrow [C'],$$

which can be shown to be a homeomorphism.

In fact, the same proof works for chamber faces and show that

Proposition 3.4.7. *For any Weyl chamber face σ_C in X , there exists a chamber face $\sigma_{C'}$ of the same dimension in X' such that*

$$d_H(\sigma_{C'}, \varphi(\sigma_C)) < +\infty.$$

As in the case of chambers, we can also define an equivalence relation on chamber faces. Two chamber faces σ_1, σ_2 in X are called equivalent if

$$d_H(\sigma_1, \sigma_2) < +\infty.$$

Clearly two equivalent chamber faces have the same dimension. Define an incidence relation on the set of equivalence classes of chamber faces $[\sigma]$ as follows: an equivalence class $[\sigma_1]$ is called a face of $[\sigma_2]$ if there are representatives $\sigma_1 \in [\sigma_1]$ and $\sigma_2 \in [\sigma_2]$ such that σ_1 is a face of σ_2 .

It can be shown that the simplicial complex with simplexes corresponding to the equivalence classes of chamber faces and the incidence relation defined here is the spherical Tits building $\Delta(G)$. Then an immediate corollary is the following:

Corollary 3.4.8. *The equivariant map $\varphi : X \rightarrow X'$ induces an isomorphism between $\Delta(G) \rightarrow \Delta(G')$.*

To finish the proof of Theorem 3.4.1, there are two cases to consider depending on whether G has any simple factor of rank equal to 1 or not.

Suppose that G has no simple factor of rank 1. Then Theorem 2.5.2 implies that G and G' are isomorphic as abstract groups. Since the maximal Furstenberg boundaries $\mathcal{X}_0, \mathcal{X}'_0$ are included canonically in $\Delta(G)$ and $\Delta(G')$ and are stable under the action of G and G' respectively and the maps $G \rightarrow \text{Aut}(\mathcal{X}_0), G' \rightarrow \text{Aut}(\mathcal{X}'_0)$ are injective, we can use the fact that $\varphi_0 : \mathcal{X}_0 \rightarrow \mathcal{X}'_0$ is a homeomorphism (Proposition 3.4.6) to show that G and G' are isomorphic as Lie groups, which proves Theorem 3.4.1 in this case.

In the second case, G is reducible and contains simple factors of rank 1. The last part of the arguments is different from above and given in [Mos, §18, Corollary 18.2]. The basic idea is as follows. Suppose G has two simple factors, $G = G_1 \times G_2$. Then the isomorphism of the Tits buildings in Proposition 3.4.8 implies that G' has a similar decomposition $G' = G'_1 \times G'_2$. Let $\pi_i : G \rightarrow G_i$ be the projection to the factors, $i = 1, 2$. Since Γ is irreducible, $\pi_i(\Gamma)$ is dense in G_i , and hence the isomorphism of $\pi_i(\Gamma)$ and $\pi_i(\Gamma')$ via the restriction φ_i of φ_0 implies that G_i is isomorphic to G'_i as Lie groups, which completes the proof of Theorem 3.4.1 in this case.

Remark 3.4.9. In both cases, the existence of the boundary value map φ_0 and the fact that it is a homeomorphism is crucial in the proof of Theorem 3.4.1. In this case, the homeomorphism (or rather isomorphism) φ_0 is induced from the quasi-isometry φ . This confirms the philosophy that when one goes to infinity, finite (or small) ambiguities are smoothed out (or ignored) and one gets nicer maps.

In the proof both the fact that φ induces an isomorphism of the Tits buildings and the fact that the boundary value map φ_0 is a homeomorphism are used. If one

uses the topological spherical buildings of the Lie groups G and G' , then φ induces an isomorphism of the topological spherical buildings, and arguments similar to those in [BuS2] should imply G and G' are isomorphic as Lie groups as in Theorem 2.5.2.

Remark 3.4.10. The proof of the Mostow strong rigidity when the rank of X is equal to 1 is very different from the higher rank case. The reason why we could not use the spherical Tits building is that there is no rigidity result for spherical Tits buildings of rank 1. Besides the original proof in [Mos1], there is at least another different proof for the rank 1 symmetric space X and one more further proof when X is a real hyperbolic space:

1. When X is of rank 1, there is a proof in [BCG1] [BCG2] using a characterization of locally symmetric spaces by the minimal entropy property and the notion of barycenter map.
2. When X is the real hyperbolic space, there is another proof due to Gromov using computation of simplicial volume of hyperbolic manifolds [BeP].

In order to mention some generalizations and simplifications of the original proof of Mostow strong rigidity for hyperbolic spaces [Mos1] [Mos2], we mention the main steps. For more details, see [GP] [Sp2]. Let $\Gamma \backslash \mathbb{H}^n$ and $\Gamma' \backslash \mathbb{H}^m$ be two compact hyperbolic manifolds, and let $\psi : \Gamma \backslash \mathbb{H}^n \rightarrow \Gamma' \backslash \mathbb{H}^m$ be an homotopy equivalence which is induced from the isomorphism on $\Gamma \cong \Gamma'$. Let $\varphi : \mathbb{H}^n \rightarrow \mathbb{H}^m$ a Γ -equivariant quasi-isometry, and its extension to $\mathbb{H}^n(\infty) \rightarrow \mathbb{H}^m(\infty)$ is denoted by $\bar{\varphi}$. The boundaries $\mathbb{H}^n(\infty)$ and $\mathbb{H}^m(\infty)$ have a canonical conformal structure.

1. The map $\bar{\varphi} : \mathbb{H}^n \rightarrow \mathbb{H}^m$ is a quasi-conformal Γ -equivariant map and hence $m = n$. (This step can also be proved by computing the cohomological dimension of Γ and Γ' from the hyperbolic manifolds $\Gamma \backslash \mathbb{H}^n$ and $\Gamma' \backslash \mathbb{H}^m$.)
2. The action of Γ on $\mathbb{H}^n(\infty)$ is ergodic. Together with the quasi-conformality of $\bar{\varphi}$, it follows that the distortion of $\bar{\varphi}$ is equal to 1 and hence $\bar{\varphi}$ is a conformal map.
3. Any Γ -equivariant conformal map $\mathbb{H}^n(\infty) \rightarrow \mathbb{H}^n(\infty)$ extends to a Γ -equivariant isometry of \mathbb{H}^n , which induces an isometry $\Gamma \backslash \mathbb{H}^n \rightarrow \Gamma' \backslash \mathbb{H}^n = \Gamma' \backslash \mathbb{H}^m$.

After this proof, there have been several generalizations and improvements which show that under suitable conditions Γ -equivariant homeomorphisms $\mathbb{H}^n(\infty) \rightarrow \mathbb{H}^n(\infty)$ are given by Möbius transformations, where Γ is not necessarily a lattice of $\text{Isom}(\mathbb{H}^n)$ (see [Tu1] [Tu2] [Iv] and the references there).

3.5 Rank rigidity of manifolds of nonpositive curvature

In a list of open problems in [Yau2], Yau raised the following question, Problem 65 (only a part of the problem is stated here in a slightly modified form):

Define the rank of manifolds of nonpositive curvature so that it agrees with the standard one for locally symmetric spaces and describe the rigidity of such nonpositively curved manifolds when the rank is at least two.

This question has generated a lot of work on nonpositively curved manifolds and was solved by Ballmann, and Burns & Spatzier (see [Bal1] [BuS1] [Bal3] [BBE] [BBS], and [Leb] [Heb] for some generalizations). The solution in [BuS1] uses classification of topological Tits buildings in Theorem 3.3.6. We briefly recall the precise formulation of the problem and its proof in [BuS1].

Let M be a complete Riemannian manifold of nonpositive (sectional) curvature. Let SM be the unit sphere subbundle in the tangent bundle of M . For each $v \in SM$, let γ_v be the geodesic in M which has the initial velocity vector v . Define the rank of v , denoted by $\text{rank}(v)$, to be the dimension of the space of parallel Jacobi fields along the geodesic γ_v . We remark that $\text{rank}(v)$ measures the maximal dimension of infinitesimal flat containing γ_v . In fact, the variation of a family of geodesics gives a Jacobi field. If the family of geodesics is parallel, the Jacobi field is parallel.

Define

$$\text{rank}(M) = \min\{\text{rank}(v) \mid v \in SM\}.$$

For any $v \in SM$, the tangent vector of γ_v is a parallel Jacobi field, and hence $\text{rank}(v) \geq 1$, which implies that

$$\text{rank}(M) \geq 1.$$

Let \tilde{M} be the universal covering space of M . If \tilde{M} is a symmetric space of noncompact type of rank r , then M is nonpositively curved. It can be shown that the $\text{rank}(M)$ as defined here is also equal to r . In fact, it is realized when v is a regular tangent vector. Hence, the definition of rank here is a correct generalization for all nonpositively curved manifolds, and solves the first part of Yau's problem.

The second part of Yau's problem on the rank rigidity was solved in [Bal3] and [BuS1].

Theorem 3.5.1. *Suppose that M is nonpositively curved and has finite volume. Then its universal covering space \tilde{M} is a space of rank 1, or a symmetric space of noncompact type, or a Euclidean space, or a product of such spaces.*

The manifold M is called irreducible if there is no finite cover of M which splits isometrically as a nontrivial product. This is related to the notion of irreducible lattices introduced before Theorem 3.4.1. It should be emphasized that \tilde{M} could split isometrically as a product.

A corollary of the above theorem is the following.

Corollary 3.5.2. *Suppose that M is nonpositively curved and has finite volume. If M is irreducible and $\text{rank}(M) \geq 2$, then M is a locally symmetric space of noncompact type.*

The basic idea of the proof in [BuS1] is as follows. Weyl simplexes in \tilde{M} are introduced using notions of regular tangent vectors. Basically, a vector is regular if it is tangent to a unique r -dimensional flat, where $r = \text{rank}(M)$. A related weaker notion of p -regular was also introduced [BBS, Definition 2.1] and ℓ -regular in [BuS1, §2]. Then Weyl simplexes at infinity are defined [BuS1, Definition 2.2] and are shown to form a compact, metric, locally connected topological spherical

Tits building of rank equal to $\text{rank}(M)$, denoted by $\Delta(\tilde{M})$ [BuS1, Proposition 3.12]. (Note that the dimension of $\Delta(\tilde{M})$ is equal to $\text{rank}(M) - 1$.) Furthermore, it is Moufang [BuS1, Proposition 3.15]. If \tilde{M} is irreducible, the building $\Delta(\tilde{M})$ is also irreducible [BuS1, Theorem 4.1].

To prove Theorem 3.5.1, it suffices to consider the case that M is irreducible and $\text{rank}(M) \geq 2$. There are two cases depending on if \tilde{M} is irreducible.

If \tilde{M} is reducible, the rigidity was proved in [Eb2, proposition 4.1]. If \tilde{M} is irreducible, then the topological building is irreducible, and hence by Theorem 3.3.6, the identity component G^0 of the topological automorphism of the topological building is a simple Lie group. Let $\Delta(G^0)$ be the associated topological spherical Tits building. Then it is isomorphic to the building $\Delta(\tilde{M})$.

This isomorphism was used to define a map

$$\Phi : \tilde{M} \rightarrow X = G^0/K,$$

where $K \subset G^0$ is a maximal compact subgroup [BuS1, p. 56]. In fact, for any $p \in \tilde{M}$, the geodesic symmetry σ_p of \tilde{M} at p defines a topological automorphism of $\Delta(\tilde{M})$, which in turn gives an involutive isomorphism θ_p of G^0 . It can be shown that θ_p induces an isometry of X and has a unique fixed point $q \in X$. Then define

$$\Phi(p) = q.$$

This idea was used earlier in the proof of a generalization of the strong Mostow rigidity by Gromov in [BGS, Chap. 4]. The last step is to show that Φ is an isometry up to a suitable multiple.

3.6 Rank rigidity for CAT(0)-spaces and CAT(0)-groups

In this section, we formulate some conjectures on how to characterize compact locally symmetric spaces and compact quotients of Euclidean buildings among CAT(0)-spaces and to characterize lattices of semisimple real and p -adic Lie groups among CAT(0)-groups. All the new results and conjectures in this subsection are based on joint work and discussion with Ralf Spatzier.

The rank rigidity of nonpositively curved manifolds in the previous subsection gave a geometric characterization of higher rank irreducible locally symmetric spaces of finite volume among all nonpositively curved complete Riemannian manifolds.

If M is a compact nonpositively curved Riemannian manifold, then its universal covering space \tilde{M} is a Hadamard manifold, i.e., a simply connected nonpositively curved Riemannian manifold. In particular, it is a CAT(0)-space. In fact, a metric space is called a *geodesic metric space* if every two points are connected by a distance minimizing (or embedded) geodesic segment. A CAT(0)-space is a geodesic metric space such that every triangle in it is thinner than a corresponding triangle in \mathbb{R}^2 with the same side lengths. Two most important examples of CAT(0)-spaces are symmetric spaces of noncompact type and Euclidean buildings.

A natural problem is to generalize the rank rigidity to compact quotients of CAT(0)-spaces, i.e., to characterize compact quotients of symmetric spaces and

Euclidean buildings among all compact quotients of CAT(0)-spaces. Another natural problem is to characterize lattices of seimsimple Lie groups among CAT(0)-groups, which are defined to be groups that *act isometrically and cocompactly on CAT(0)-spaces*.

To formulate conjectures, we recall several notions. A geodesic metric space is called *geodesically complete* if every geodesic segment can be extended infinitely in both directions. We note that every convex subspace of \mathbb{R}^n or of any CAT(0)-space is also a CAT(0)-space but any such bounded convex subspace is not geodesically complete. A metric space is called *irreducible* if no finite covering space splits as a product isometrically.

To generalize the rank rigidity to CAT(0)-spaces, the first problem is to define the notion of rank. As explained in [Bal1, Problem 1 in Introduction], “assume that X is a geodesically complete and locally compact Hadamard space”, which is equivalent to that X is a geodesically complete locally compact CAT(0)-space, *Problem 1* is to define the rank of X in such a way that under the assumption that the isometry group of X satisfies the duality condition, then

1. the rank k of X is greater than or equal to 2 if and only if every geodesic of X is contained in a k -flat.
2. the rank k of $X = 1$ implies some (non-uniform) hyperbolicity of the geodesic flow.

Problems 2 and 3 of [Bal1, Introduction] are closely related and are stated as follows:

1. *Problem 2.* Assume that X is irreducible and that the group of isometries of X satisfies the duality condition. Show that X is a symmetric space or a Euclidean building if every geodesic of X is contained in a k -flat, $k \geq 2$.
2. *Problem 3.* Assume that Γ is a cocompact and properly discontinuous group of isometries of X . Show that Γ satisfies the duality condition.

There have seen several different candidates for definition of the rank. According to [BaB1, p. 873], the condition that X has rank greater than or equal to 2 could mean one of the following:

1. every geodesic belongs to a flat plane;
2. every geodesic bounds a flat half plane;
3. every geodesic is a side of a flat strip;
4. every finite geodesic segment is a side of a flat strip.

Clearly, the condition becomes weaker down the list. In [BaB1], under the weakest of the above conditions, it was shown that if X is a simply connected, geodesically complete 3-dimensional Euclidean polyhedron of nonpositive curvature of rank at least 2 and admits a compact quotient, then X is either a Riemannian product or a thick Euclidean building of type \tilde{A}_3 or \tilde{B}_3 . A similar result holds for 2-dimensional case too [BaB2] [Bar].

In [BaBu], the rank of a geodesic in X is defined to be 1 if it does not bound a half plane, and hence that a geodesic is of rank at least 2 if it bounds a flat half plane.

The purpose of this subsection is to define a slightly different notion of rank and formulate two rigidity conjectures.

Given a geodesic metric space M whose universal covering space \tilde{M} is a CAT(0)-space. We define its *rank* of \tilde{M} , also called the *rank* of M , to be the maximum of integer r such that for every geodesic γ in \tilde{M} , there exists a flat strip in \tilde{M} whose center is γ , i.e., for some $\varepsilon > 0$, there exists an isometric embedding $\gamma \times (-\varepsilon, \varepsilon)^{r-1} \rightarrow \tilde{M}$ such that $\gamma \times \{(0, \dots, 0)\}$ is equal to γ . Note that $\gamma \times (-\varepsilon, \varepsilon)^{r-1} \cong \mathbb{R} \times (-\varepsilon, \varepsilon)^{r-1}$. When $r = 2$, it is a flat strip of dimension 2.

For Riemannian manifolds of nonpositively sectional curvature, the rank is defined to be the dimension of parallel Jacobian fields, which represent infinitesimal flat strips around the geodesic. The above definition seems to be a natural generalization to the setup of CAT(0)-spaces.

Conjecture 3.6.1. *Let M be a compact irreducible geodesic metric M satisfying the conditions:*

1. *The space M is geodesically complete and its rank is at least 2*
2. *The universal covering space \tilde{M} is a CAT(0)-space.*

Then \tilde{M} is isometric to products of symmetric spaces of noncompact and Euclidean buildings.

Remark 3.6.2. It is clear that this conjecture is a variant of Problems 1, 2 and 3 of [Ba1, Introduction] mentioned above. The difference is in the definition of rank. There is also a rigidity conjecture in [BaBu, Conjecture B] in terms of diameter of the Tits metric on the geodesic sphere at infinity.

The first observation is that the following result is true.

Proposition 3.6.3. *Under the assumption of Conjecture 3.6.1, if the rank of \tilde{M} is equal to the dimension of \tilde{M} , then every geodesic in \tilde{M} is contained in a r -flat, where r is the rank of \tilde{M} .*

Proof. We prove this by contradiction. If not, there exists a geodesic γ in \tilde{M} such that γ is contained in a r -flat strip $F = \gamma \times (-\varepsilon, \varepsilon)^{r-1}$, where ε is the maximum value. Now we claim that this flat strip can be enlarged under the assumption that $\dim \tilde{M} = r$. For every geodesic γ' on the boundary of the strip $\gamma \times (-\varepsilon, \varepsilon)^{r-1}$ that is parallel to γ , γ' is contained in a r -flat strip $F' = \gamma' \times (-\varepsilon', \varepsilon')^{r-1}$. Take a geodesic γ'' in $\gamma' \times (-\varepsilon', \varepsilon')^{r-1}$ different from γ' but is parallel to γ' . Then the distance between γ and γ'' is bounded and hence they are parallel and contained in a flat 2-dimensional strip F'' since \tilde{M} is a CAT(0)-space. Since the dimension of \tilde{M} is equal to the flat strip and \tilde{M} is a CAT(0)-space, the flat strip F'' consists of two pieces, where the first piece F''_1 is contained in F and the second part is contained in F' . Furthermore, when γ' changes among all possible boundary geodesics parallel to γ , and γ'' also changes in the flat strip F' , then the union of

the first piece F_1'' is equal to F . This means that the union of the 2-dimensional flat strip F'' is a flat strip that contains F . Since the set of directions at any point in \tilde{M} (or rather set of geodesic rays in \tilde{M} out of any point) is compact, this union of F'' strictly contains F . This implies that F is not the maximal flat strip containing γ .

Remark 3.6.4. The above proposition supports the conjecture that the assumption that $\dim \tilde{M}$ is equal to the rank of \tilde{M} in the proposition can be dropped. If this is the case, the definition of rank here will give a solution to Problem 1 of [Ball, Introduction] mentioned before.

The following result is due to Kleiner (see [Bar]).

Proposition 3.6.5. *Under the above assumption of M in Conjecture 3.6.1, if the rank r of \tilde{M} is at least 2 and equal to the dimension of \tilde{M} and every geodesic in \tilde{M} is contained in a r -flat, then \tilde{M} is a product of Euclidean buildings.*

Corollary 3.6.6. *Under the above assumption of M in Conjecture 3.6.1, if the rank of M is equal to the dimension of M , then \tilde{M} is a product of Euclidean buildings.*

Remark 3.6.7. A natural approach to prove Conjecture 3.6.1 is to use the approach of [BuS1] and follow the steps:

1. Use the assumption that \tilde{M} is geodesically complete, the rank of \tilde{M} is at least 2 and M is irreducible to show that the geodesic boundary $\tilde{M}(\infty)$ has the structure of a Tits building. To use a construction similar to that in [Bus1], we need the notion of regular geodesics, which seems difficult. The work in [CL] might be helpful to show that $\tilde{M}(\infty)$ has the structure of a Tits building.
2. Once we have Step 1, then the characterization of symmetric spaces and Euclidean buildings in terms of a Tits building structure of the geodesic boundary in [Leb] finishes the proof.

Remark 3.6.8. The rank rigidity for nonpositively curved manifolds holds for finite volume Riemannian manifolds. On the other hand, in Conjecture 3.6.1 above, the assumption that M is compact is probably necessary in view of the example mentioned in [BaB1, Theorem 1.2], which states that there exists a 3-dimensional geodesically complete simply connected Euclidean polyhedron of nonpositive curvature in which every geodesic is contained in a flat plane but which is not a thick Euclidean building. By [BaB1, Theorem 1.1], such a space does not admit any compact quotient under a discrete isometry group.

Recall that a group Γ is said to be irreducible if no finite index subgroup is isomorphic to product of two infinite groups. As recalled earlier, a CAT(0)-group is a group that acts properly and isometrically on a CAT(0)-space with a compact quotient. Since symmetric spaces of noncompact type are nonpositively curved and simply connected, they are CAT(0)-spaces. As mentioned before, Euclidean

buildings are also CAT(0)-spaces. It follows that lattices acting cocompactly on symmetric spaces of noncompact type or Euclidean buildings are CAT(0)-groups. It is also clear that the fundamental group of any compact nonpositively curved Riemannian manifold is a CAT(0)-group. See [BH] for examples and references.

Motivated by the rigidity results for nonpositively curved spaces, the following conjecture seems reasonable. This problem of characterizing lattices acting on symmetric spaces and buildings among CAT(0)-groups is certainly natural and important.

Conjecture 3.6.9. *Assume that Γ is an irreducible torsion-free CAT(0)-group such that every nontrivial element of Γ is contained in an abelian subgroup of rank at least 2. Then Γ is isomorphic to a lattice in the isometry group of symmetric spaces, Euclidean buildings, or their products.*

3.7 Classification of isoparametric submanifolds

Another application of the topological spherical Tits buildings to differential geometry concerns isoparametric submanifolds in Euclidean spaces [Tho2] [Ter1].

Isoparametric hypersurfaces in \mathbb{R}^{n+1} were introduced by E.Cartan and defined to be hypersurfaces in \mathbb{R}^{n+1} whose principal curvatures are constant. Clearly, affine n -planes and n -spheres in \mathbb{R}^{n+1} are isoparametric. He showed that any connected isoparametric hypersurface is contained an affine n -plane, or n -sphere, or the product of a k -sphere with an affine $(n - k)$ -plane.

Isoparametric submanifolds of higher codimension were introduced by Terng, motivated by the problem of finding submanifolds with simple local invariants. They turn out to be related to Coxeter groups and spherical Tits buildings, which reflect their global symmetries.

Definition 3.7.1. *A submanifold in \mathbb{R}^{n+r} of codimension r is called isoparametric if its normal bundle is flat and the principal curvatures in the direction of any parallel normal vector field are constant.*

See [Ter1] [Tho1] and the references there for the history and more motivations. (There is a thorough discussion about the origin of isoparametric hypersurfaces and related topics in [Tho1].)

It is known that any compact isoparametric submanifold is contained in some sphere, and hence the only compact isoparametric hypersurfaces are spheres.

Isoparametric submanifolds can be defined explicitly in terms of isoparametric maps and the associated foliation (or disjoint decomposition) as follows.

Definition 3.7.2. *A smooth map $f = (f_1, \dots, f_r) : \mathbb{R}^{n+r} \rightarrow \mathbb{R}^r$ is called isoparametric if*

1. *f has a regular value,*
2. *for any $i, j, k = 1, \dots, r$, $\langle \nabla f_i, \nabla f_j \rangle$ and Δf_k are constants on all fibers of the map,*
3. *for any $i, j = 1, \dots, r$, on each fiber of f , $[\nabla f_i, \nabla f_j]$ is a linear combination of $\nabla f_1, \dots, \nabla f_r$ with constant coefficients.*

For any isoparametric map f , the fibers of f give a disjoint decomposition, called an *isoparametric foliation*, of \mathbb{R}^{n+r} . It is known that the fiber over a regular value in \mathbb{R}^r is an isoparametric submanifold of \mathbb{R}^{n+r} . Conversely, it is also known [Ter2] that every isoparametric submanifold occurs as a leaf in an isoparametric foliation.

An isoparametric submanifold is called *full* if it is not contained in a proper affine subspace, and *irreducible* if it is not the product of two isoparametric submanifolds. Since a complete isoparametric submanifold is the product of a compact isoparametric submanifold with a Euclidean space, we will only discuss compact, full, and irreducible isoparametric submanifolds.

Another important class of isoparametric submanifolds arises from group spheres in the theory of symmetric spaces. Specifically, let G be a noncompact semisimple Lie group and $K \subset G$ a maximal compact subgroup. Let $X = G/K$ be the associated symmetric space, and r be the rank of X . Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the Cartan decomposition of \mathfrak{g} determined by the Cartan involution θ corresponding to K . Then the tangent space $T_{x_0}X$ of X at the basepoint $x_0 = K$ can be identified with \mathfrak{p} . Since the adjoint action of K preserves the Cartan decomposition, it restricts to the subspace \mathfrak{p} . This action of K on $\mathfrak{p} = T_{x_0}X$ preserves the inner product and is called the isotropy representation of the symmetric space X . The K -orbits in \mathfrak{p} give an isoparametric foliation, and the principal K -orbits, i.e., the orbits of maximal dimension, are isoparametric submanifolds. They are called flag manifolds. See [Ter1] for detailed discussion of their geometry as submanifolds of \mathfrak{p} .

Remark 3.7.3. When the rank r of X is equal to 1, every nontrivial K -orbit is a principal orbit and is a sphere. In general, the principal K -orbits are of codimension r . Since the exponential map $\exp : \mathfrak{p} \rightarrow X$ is a K -equivariant diffeomorphism, these K -orbits in \mathfrak{p} can be looked upon as submanifolds in X . Then they are called group spheres in [GJT] for the following reason. Harmonic functions on \mathbb{R}^n are characterized by the property that the average value on any sphere is equal to the value at the center. For the symmetric space, a function is called strongly harmonic if it is a joint eigenfunction of all invariant differential operators, which form a ring with r generators. A strongly harmonic function on X is characterized by the property that its average value over any group sphere is equal to its value at the center. The principal K -orbits can also be identified with the Poisson boundary of X , which is crucial for the generalization of the Poisson integral formula by Furstenberg [Fu1].

The principal orbits of the isotropy representations of symmetric spaces are the main sources of isoparametric submanifolds in Euclidean spaces.

Proposition 3.7.4. *Any homogeneous isoparametric submanifold in a Euclidean space is a principal K -orbit of the isotropy representation of a symmetric space $X = G/K$.*

This result was proved by Palais and Terng [PT]. The next result of Thorbergsson [Tho2] removed the homogeneity assumption when the codimension is greater than or equal to 3.

Theorem 3.7.5. *Any compact irreducible, full isoparametric submanifold in a Euclidean space of codimension at least 3 is a principal orbit of the isotropy representation of a symmetric space.*

It should be pointed out that the assumption on the codimension being at least 3 is necessary. There are many inhomogeneous examples of isoparametric submanifolds in Euclidean spaces of codimension 2. As pointed out earlier, in the case of dimension 1, compact isoparametric submanifolds are spheres. So the classification is only open in the case of codimension 2 (see [FKM], and also [Kr1] for recent progress on this problem). Theorem 3.7.5 has later been proved without using buildings in [HeL], [Ol] and [EsH] (see also [Esc1] [Esc2]). But the connection with the building is beautiful and fits in well with the themes of this survey.

The idea of the proof of Theorem 3.7.5 is as follows. Let M^n be a compact, full and irreducible isoparametric submanifold in \mathbb{R}^{n+r} , where $r \geq 3$. Assume without loss of generality that M is contained in the unit sphere S^{n+r-1} (with center at the origin).

1. Associate a simplicial complex $\Delta(M)$ of dimension r to M whose underlying space is the unit sphere S^{n+r-1} .
2. Show that the simplicial complex $\Delta(M)$ is a spherical Tits building of rank r , which is then a topological spherical building due to its realization in S^{n+r-1} .
3. Apply Theorem 3.3.6 on classification of topological Tits buildings to show that $\Delta(M)$ is the topological building associated with a simple Lie group G . In fact, the assumption $r \geq 3$ implies that the building is Moufang.
4. The symmetry of \mathbb{R}^{n+r} with respect to the origin picks out a maximal compact subgroup K of G . Define an action of K on \mathbb{R}^{n+r} using the fact that the underlying space of $\Delta(M)$ is the unit sphere S^{n+r-1} .
5. Show that the action of K on \mathbb{R}^{n+r} is the isotropy representation of the symmetric space G/K and its principal orbits are isometric to the isoparametric submanifold M up to scaling.

In this proof, the use of the topological building, rather than the usual building, is important to get Lie groups G and K . The building is also crucial to get the desired action of K on \mathbb{R}^{n+r} .

To motivate the construction of the simplicial complex $\Delta(M)$ in Step (1), we explain a construction of the spherical Tits building $\Delta(G)$ using the geometry of a principal K -orbit in the isotropy representation.

Let $H \in \mathfrak{p}$ be a regular unit vector, i.e., a unit vector contained in a unique maximal abelian subspace in \mathfrak{p} . Then the orbit $K \cdot H$ is a principal orbit, denoted by \mathcal{O} . Let $N\mathcal{O}$ be its normal bundle in \mathfrak{p} . Then for any $p \in \mathcal{O}$, there exists a

unique maximal abelian subalgebra \mathfrak{a} in \mathfrak{p} which contains p . Furthermore, \mathfrak{a} can be identified with the fiber $N_p\mathcal{O}$ in such a way such that the origin of the vector space $N_p\mathcal{O}$ corresponds to $p \in \mathfrak{a}$. In fact, since \mathcal{O} is contained in the unit sphere in \mathfrak{p} , p is contained in $N_p\mathcal{O}$; the Cartan decomposition $G = K \exp \mathfrak{a}K$ implies that $X = K \exp \mathfrak{a} \cdot x_0$, where $x_0 = K \in X$, and hence the codimension of \mathcal{O} is equal to $\dim \mathfrak{a}$, the rank of G or X .

The Lie algebra \mathfrak{g} decomposes into root spaces

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y, H \in \mathfrak{a}\},$$

and $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\}$. Each root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ defines a root hyperplane \mathbf{H}_α , and the root hyperplanes divide \mathfrak{a} into Weyl chambers and chamber faces. The intersection of the unit sphere S^{n+r-1} with these chambers and chamber faces determines a finite simplicial complex, the Coxeter complex associated with G . When $p \in \mathcal{O}$ changes, \mathfrak{a} changes, and these Coxeter complexes fit together into the spherical Tits building $\Delta(G)$.

The crucial observation here is that these Coxeter complexes can be constructed from the geometry of \mathcal{O} as a submanifold. In fact, the root hyperplanes \mathbf{H}_α in \mathfrak{a} are exactly the intersection of \mathfrak{a} with the set of focal points of $\mathcal{O} \in \mathfrak{p}$, i.e., the set of critical values of the exponential map restricted to the normal bundle $N\mathcal{O}$. In the case of rank 1, for example, $G = SO(2, 1)$, \mathcal{O} is the unit sphere, and the only focal point is the origin; each maximal abelian subspace \mathfrak{a} has dimension 1, and the only root hyperplane is the origin. This example also explains the need for the shift in identifying \mathfrak{a} with $N_p\mathcal{O}$ above. (To get a less trivial but explicit example, think of the product $SO(2, 1) \times SO(2, 1)$. The Weyl chamber decomposition is given by the coordinate quadrants.)

Now the construction of the simplicial complex $\Delta(M)$ in Step (1) is similar. Let NM be the normal bundle of M in \mathbb{R}^{n+r} . For each point $p \in M$, the intersection of the set of focal points of M in \mathbb{R}^{n+r} with the fiber N_pM , which is identified with a subspace of \mathbb{R}^{n+r} by shifting by p , gives a collection of hyperplanes. They divide N_pM into chambers and chamber faces, whose intersections with the unit sphere S^{n+r-1} give a finite simplicial complex. When $p \in M$ changes, these finite complexes form the simplicial complex $\Delta(M)$.

To show that $\Delta(M)$ is a building is not easy. In fact, in the above construction of $\Delta(G)$ using the principal K -orbit \mathcal{O} , \mathfrak{a} corresponds to a maximal totally geodesic subspace in X passing through the basepoint $x_0 = K$, and its Coxeter complex is an apartment in $\Delta(G)$. They form a system of apartments in $\Delta(G)$ which is invariant under K but not under G . This causes some difficulties in showing $\Delta(G)$ satisfies the axioms of buildings, for example, that any two chambers are contained in an apartment. In general, given two chambers in $\Delta(G)$, apartments containing the two chambers are not an apartment corresponding to a flat in X passing through x_0 , or corresponding to \mathfrak{a} .

This shows that $\Delta(M)$ is defined as the union of special apartments. Proving that $\Delta(M)$ is a spherical Tits building in Step (2) is not easy. In fact, it is the

major part of the paper [Tho2], which depends on a local approach of Tits on constructing buildings.

Remark 3.7.6. Though isoparametric submanifolds of codimension 2 have not been classified, many known examples are connected to rank-2 buildings. In fact, in [FKM], an isoparametric hypersurface in a sphere with precisely four principal curvatures was constructed for every real representation of a Clifford algebra. It is proved in [Tho3] that each of these hypersurfaces is the flag manifold of a (differentiable) polar plane, i.e., a building of type C_2 , also called a generalized quadrangle. Most of these polar planes are not Moufang buildings, and hence Theorem 3.3.6 does not apply. In fact, they do not come from isotropy representations of symmetric spaces. See [Kr1] for related results on isoparametric submanifolds of codimension 2. Together with Theorem 3.7.5, this result shows that all presently known isoparametric submanifolds are associated with buildings.

Remark 3.7.7. Recently, topological buildings have been used in independent, closely related works of Lytchak and of Fang, Grove and Thorbergsson to classify isoparametric foliation of non-negatively curved compact symmetric spaces and positively curved Riemannian manifolds. More specifically, the result of Lytchak can be stated as follows: *Let M be a non-negatively curved irreducible symmetric space. Let \mathcal{F} be a polar foliation on M (also called isoparametric foliation, or singular Riemannian foliation with sections). Then either the foliation is hyperpolar (i.e. the sections are flat), or the symmetric space has rank 1, or, possibly, the codimension of the foliation is two.* The result of Fang, Grove and Thorbergsson can be stated as follows: *Let M be a simply connected positively curved Riemannian manifold, and let G be a compact Lie group acting on M in a polar fashion. If the cohomogeneity of the action is at least 3, then M is equivariantly diffeomorphic to a linear polar G -action on a rank one symmetric space, i.e., sphere, complex or quaternionic projective spaces.*

3.8 Spherical buildings and compactifications of locally symmetric spaces

Let $\mathbf{G} \subset GL(n, \mathbb{C})$ be a linear semisimple algebraic group defined over \mathbb{Q} , $G = \mathbf{G}(\mathbb{R})$ its real locus, a Lie group with finitely many components. Let $K \subset G$ be a maximal compact subgroup as usual, and $X = G/K$ the associated Riemannian symmetric space of noncompact type.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup, i.e., a subgroup commensurable to $\mathbf{G}(\mathbb{Q}) \cap GL(n, \mathbb{Z})$. For simplicity, assume that Γ is torsion free or even neat in the following (see [Bo] for the definition of a neat arithmetic subgroup). Then $\Gamma \backslash X$ is a locally symmetric space of finite volume.

Let r be the \mathbb{Q} -rank of \mathbf{G} , which is defined to be the maximal dimension of \mathbb{Q} -split tori contained in \mathbf{G} . It is known that $\Gamma \backslash X$ is non-compact if and only if the \mathbb{Q} -rank r of \mathbf{G} is positive. In the following, we assume that $r > 0$, and hence $\Gamma \backslash X$ is noncompact.

Remark 3.8.1. A good example to keep in mind is $\mathbf{G} = SL(n)$, $G = SL(n, \mathbb{R})$, $K = SO(n)$, and $X = SL(n, \mathbb{R})/SO(n)$ is the space of positive definite quadratic

forms of determinant 1. The \mathbb{Q} -rank of $\mathbf{G} = SL(n)$ is equal to $n - 1$, and finite index subgroups $\Gamma \subset SL(n, \mathbb{Z})$ are arithmetic subgroups of $SL(n, \mathbb{Q})$.

Let $\Delta_{\mathbb{Q}}(\mathbf{G})$ be the spherical Tits building of the algebraic group \mathbf{G} defined over \mathbb{Q} , i.e., its simplexes correspond to \mathbb{Q} -parabolic subgroups. In this building, apartments can be described as follows. For any maximal \mathbb{Q} -split torus $\mathbf{T} \subset \mathbf{G}$, there are only finitely many \mathbb{Q} -parabolic subgroups of \mathbf{G} that contain \mathbf{T} . The simplexes corresponding to them form an apartment, and any apartment is of this form.

Then $\mathbf{G}(\mathbb{Q})$ and hence Γ acts simplicially on $\Delta_{\mathbb{Q}}(\mathbf{G})$. Note that for any element $g \in \mathbf{G}(\mathbb{Q})$ and $x \in \Delta_{\mathbb{Q}}(\mathbf{G})$, if g fixes x , then g acts as the identity map on the unique simplex which contains x as an interior point. By the reduction theory of arithmetic groups (see [Bo] [BJ1] and references there), there are only finitely many Γ -conjugacy classes of \mathbb{Q} -parabolic subgroups and Γ acts, the quotient $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$ is a finite simplicial complex.

The basic theme of this and the next subsection is that the geometry at infinity and compactifications of $\Gamma \backslash X$ are closely related to $\Delta_{\mathbb{Q}}(\mathbf{G})$ and $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$. For example, when $r = 1$, $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$ consists of finitely many points, which are in canonical one-to-one correspondence with the ends of $\Gamma \backslash X$.

Borel-Serre compactification

A well-known compactification of $\Gamma \backslash X$ is the Borel-Serre compactification in [BoS2]. The compactification has also been discussed in many papers, for example, [GHM] [RS] [Ro2] [Gra1] [Gra2] [JM] [BJ1] [BJ2]. We will outline the modified approach in [JM] [BJ1] [BJ2].

It is constructed in the following three steps:

1. For every \mathbb{Q} -parabolic subgroup \mathbf{P} of \mathbf{G} , choose a boundary component $e(\mathbf{P})$.
2. Attach all the boundary components $e(\mathbf{P})$ to X at infinity to form a partial compactification⁸ $X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$.
3. Show that the Γ -action on X extends to a continuous action on $X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$ with a compact quotient $\Gamma \backslash X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$.

In this general approach, the Langlands decomposition of \mathbb{Q} -parabolic subgroups and the induced horospherical decomposition of X play a fundamental role.

For each \mathbb{Q} -parabolic subgroup \mathbf{P} , let $P = \mathbf{P}(\mathbb{R})$ be its real locus. Let $N_{\mathbf{P}}$ be the unipotent radical of P . Let θ be the Cartan involution associated with the maximal compact subgroup K . Then there are two subgroups $A_{\mathbf{P}}$ and $M_{\mathbf{P}}$ of P which are stable under θ and

$$M_{\mathbf{P}} = \cap_{\chi \in X_{\mathbb{Q}}(\mathbf{G})} \{g \in G \mid |\chi(g)| = 1\},$$

where $X_{\mathbb{Q}}(\mathbf{G})$ is the set of characters of \mathbf{G} defined over \mathbb{Q} , such that

$$P = N_{\mathbf{P}} A_{\mathbf{P}} M_{\mathbf{P}} \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times M_{\mathbf{P}}. \quad (3.8.1)$$

⁸By a partial compactification of X , we mean any space Y which contains X as an open dense subset. The reason is that Y is obtained by compactifying X in some directions or parts. Some other names for such spaces are bodifications and enlargements.

The component $A_{\mathbf{P}}$ is diffeomorphic to its Lie algebra and called the \mathbb{Q} -split component. The product $A_{\mathbf{P}}M_{\mathbf{P}}$ is the Levi component of P . The decomposition of P in Equation 3.8.1 is called the \mathbb{Q} -Langlands decomposition of $P = \mathbf{P}(\mathbb{R})$, since it depends on the \mathbb{Q} -structure of \mathbf{P} . In general, it is different from the (\mathbb{R} -)Langlands decomposition of P recalled in Equation 3.2.1. In fact, $A_{\mathbf{P}} \subset A_P$ in general.

Define

$$X_{\mathbf{P}} = M_{\mathbf{P}}/K \cap M_{\mathbf{P}},$$

called the \mathbb{Q} -boundary symmetric space of \mathbf{P} . Since P acts transitively on X , then the \mathbb{Q} -Langlands decomposition of P induces the \mathbb{Q} -horospherical decomposition

$$X = N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}.$$

Let X_P be the \mathbb{R} -boundary symmetric space in Equation 3.2.2. Then $X_{\mathbf{P}}$ is in general the product of X_P , which is a symmetric space of noncompact type, with a Euclidean space.

Now the Borel-Serre compactification of $\Gamma \backslash X$ can be constructed as follows. For each \mathbb{Q} -parabolic subgroup \mathbf{P} , define its boundary component $e(\mathbf{P})$ by

$$e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}.$$

It is attached at the infinity of X using the \mathbb{Q} -horospherical decomposition with respect to P . In fact, the attachment is achieved when the $A_{\mathbf{P}}$ -component goes to infinity through the positive chamber and stays further and further away from its walls.

Then the Borel-Serre partial compactification \overline{X}^{BS} of X is defined to be

$$\overline{X}^{BS} = X \cup \coprod_{\mathbf{P}} N_{\mathbf{P}} \times X_{\mathbf{P}}$$

with a suitable topology. In fact, as mentioned above, the convergence of interior points to the boundary points in X_P is described in terms of the \mathbb{Q} -horospherical decomposition of X with respect to \mathbf{P} . For any two \mathbb{Q} -parabolic subgroups \mathbf{P} and \mathbf{Q} , $e(\mathbf{P})$ is contained in the closure of $e(\mathbf{Q})$ if and only if $\mathbf{P} \subseteq \mathbf{Q}$; and the convergence sequences of points in $e(\mathbf{Q})$ to points is described in terms of a relative \mathbb{Q} -horospherical decomposition.

The Γ -action on X extends to a continuous action on \overline{X}^{BS} . Since the boundary components $e(\mathbf{P})$ are sufficiently large, $\Gamma \cap P$ acts properly on $e(\mathbf{P})$, and Γ also acts properly on \overline{X}^{BS} . Using the reduction theory of arithmetic groups, it can be shown that the quotient $\Gamma \backslash \overline{X}^{BS}$ is a compact Hausdorff space, which is defined to be the Borel-Serre compactification of $\Gamma \backslash X$, also denoted by $\overline{\Gamma \backslash X}^{BS}$.

If Γ is torsion-free, then Γ acts freely on \overline{X}^{BS} , and the quotient $\overline{\Gamma \backslash X}^{BS}$ is a compact real analytic manifold with corners.

When $\mathbf{G} = SL(2)$, $\Gamma \backslash X$ is a Riemann surface with finitely many cusps, and $\overline{\Gamma \backslash X}^{BS}$ is obtained by adding one (horo)-circle to each cusp of $\Gamma \backslash X$. In this case, $\overline{\Gamma \backslash X}^{BS}$ is a real analytic manifold with boundary, adding one circle at each end of $\Gamma \backslash X$.

An important property of $\overline{\Gamma \backslash X}^{BS}$ is that when Γ is torsion-free, the inclusion $\Gamma \backslash X \rightarrow \overline{\Gamma \backslash X}^{BS}$ is a homotopy equivalence. In fact, $\overline{\Gamma \backslash X}^{BS}$ is a manifold with corners with the interior equal to $\Gamma \backslash X$. Since $\Gamma \backslash X$ is a $K(\Gamma, 1)$ -space, $\overline{\Gamma \backslash X}^{BS}$ is also a $K(\Gamma, 1)$ -space in this case. Since $\overline{\Gamma \backslash X}^{BS}$ is a compact real analytic manifold with corners and hence admits a finite triangulation, it follows that $\overline{\Gamma \backslash X}^{BS}$ is a finite $K(\Gamma, 1)$ -space. Equivalently, we have the following result.

Proposition 3.8.2. *The partial Borel-Serre compactification \overline{X}^{BS} is contractible and is a model of $E\Gamma$ when Γ is torsion-free. The boundary $\partial \overline{X}^{BS}$ is homotopy equivalent to the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ and hence is equivalent to a bouquet of spheres of dimension $r - 1$, where r is the \mathbb{Q} -rank of \mathbf{G} .*

Proof. The contractibility follows from the fact that \overline{X}^{BS} is a manifold with corners and the interior equal to X and the contractibility of X . Since Γ acts properly on \overline{X}^{BS} and Γ is torsion-free, it follows that \overline{X}^{BS} is a model of $E\Gamma$. Since the boundary components $e(\mathbf{P})$ are contractible and satisfies the inclusion relation $e(\mathbf{P}) \subset e(\mathbf{Q})$ if and only if $\mathbf{P} \subset \mathbf{Q}$, i.e., the inclusion relation is the opposite of the inclusion relation in the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, it follows that the boundary $\partial \overline{X}^{BS}$ is homotopy equivalent to the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$.

The Borel-Serre compactification $\overline{\Gamma \backslash X}^{BS}$ is important in studying the cohomology groups of Γ . Its applications to the cohomology groups of Γ have been discussed in many articles and surveys [Se2] [RS] [Br1] [Br3] [Ro2], and we will not discuss them except mentioning that the cohomological dimension of Γ is equal to $\dim \Gamma \backslash X - r$, where r is the \mathbb{Q} -rank of \mathbf{G} . In proving this result, the fact that the boundary $\partial \overline{X}^{BS}$ is a cell complex homotopy equivalent to the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ was used.

Tits compactification

In [JM], the general approach outlined above to compactify $\Gamma \backslash X$ was used to construct a partial compactification \overline{X}^T whose boundary is exactly the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, and the quotient $\Gamma \backslash \overline{X}^T$ is a compactification whose boundary is the finite simplicial complex $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$. Due to the relation to the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, this compactification of $\Gamma \backslash X$ was called the Tits compactification in [JM].

Briefly, the Tits compactification is constructed as follows. Let \mathbf{G} be a linear semisimple algebraic group defined over \mathbb{Q} as above. For each \mathbb{Q} -parabolic subgroup \mathbf{P} , let $\mathfrak{a}_{\mathbf{P}}$ be the Lie algebra of $A_{\mathbf{P}}$. The parabolic subgroup \mathbf{P} determines a positive chamber $\mathfrak{a}_{\mathbf{P}}^+$. Let $\mathfrak{a}_{\mathbf{P}}^+(\infty)$ be the set of unit vectors in $\mathfrak{a}_{\mathbf{P}}^+$,

$$\mathfrak{a}_{\mathbf{P}}^+(\infty) = \{H \in \mathfrak{a}_{\mathbf{P}}^+ \mid \|H\| = 1\},$$

where $\|\cdot\|$ is the restriction of the Killing form to $\mathfrak{a}_{\mathbf{P}}$. Clearly $\mathfrak{a}_{\mathbf{P}}^+(\infty)$ is an open simplex. Define the boundary component $e(\mathbf{P})$ by

$$e(\mathbf{P}) = \mathfrak{a}_{\mathbf{P}}^+(\infty).$$

We also attach the boundary components $\mathfrak{a}_{\mathbf{P}}^+(\infty)$ to the infinity of X through the \mathbb{Q} -horospherical decomposition associated with \mathbf{P} , and obtain the partial Tits compactification

$$\overline{X}^T = X \cup \coprod_{\mathbf{P}} \mathfrak{a}_{\mathbf{P}}^+(\infty).$$

It can be checked easily that $\mathfrak{a}_{\mathbf{P}}^+(\infty)$ is the interior of the simplex in the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, and hence $\coprod_{\mathbf{P}} \mathfrak{a}_{\mathbf{P}}^+(\infty)$ can be identified with $\Delta_{\mathbb{Q}}(\mathbf{G})$, and hence

$$\overline{X}^T \cong X \cup \Delta_{\mathbb{Q}}(\mathbf{G}).$$

It can be shown that Γ acts continuously with a compact quotient $\Gamma \backslash \overline{X}^T$, which is also denoted by $\overline{\Gamma \backslash X}^T$ and whose building is $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$.

3.9 Geodesic compactification, Gromov compactification and large scale geometry

The Tits compactification $\overline{\Gamma \backslash X}^T$ is closely related to the structure of geodesics in $\Gamma \backslash X$ and the large scale geometry of $\Gamma \backslash X$. In fact, it is homeomorphic to the geodesic compactification $\Gamma \backslash X \cup \Gamma \backslash X(\infty)$ and the Gromov compactification $\overline{\Gamma \backslash X}^G$, and its boundary $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$ is also the base of the cone of the tangent space at infinity $T_{\infty} \Gamma \backslash X$. Furthermore, the cone over $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$ is within a finite Gromov-Hausdorff distance of $\Gamma \backslash X$, and this result is closely related to a conjecture of Siegel.

Geodesic compactification $\Gamma \backslash X \cup \Gamma \backslash X(\infty)$

In the rest of this subsection, X denotes the symmetric space G/K of non-compact type, and a general Riemannian manifold is denoted by M . As pointed out earlier, X is simply connected and nonpositively curved, and hence the set of equivalence classes of geodesics defines the sphere at infinity $X(\infty)$, which is the boundary of the geodesic compactification $X \cup X(\infty)$.

On the other hand, since $\Gamma \backslash X$ is noncompact, there are geodesics which are bounded, or geodesics which are unbounded but do not go to infinity. It is necessary to choose a suitable class of geodesics. In [JM], EDM geodesics were used to define the geodesic compactification $\Gamma \backslash X \cup \Gamma \backslash X(\infty)$.

Specifically, a (unit speed, directed) geodesic $\gamma(t)$ in $\Gamma \backslash X$ is called eventually distance minimizing (EDM) if for all $t_1, t_2 \gg 0$,

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|.$$

Clearly, every EDM geodesic $\gamma(t)$ goes to infinity of $\Gamma \backslash X$, i.e., leaves every compact subset of $\Gamma \backslash X$, as $t \rightarrow +\infty$. For $t_0 \gg 0$, the restriction $\gamma : [t_0, +\infty) \rightarrow \Gamma \backslash X$ is an isometric embedding and is sometimes called a ray in $\Gamma \backslash X$. Two EDM geodesics γ_1, γ_2 are called *equivalent* if

$$\limsup_{t \rightarrow +\infty} d(\gamma_1(t), \gamma_2(t)) < +\infty.$$

Let $\Gamma \backslash X(\infty)$ be the set of equivalence classes of EDM geodesics in $\Gamma \backslash X$.

EDM geodesics were studied in [Ha1] [Ha2] [Le2], and were completely classified in [JM]. In particular, $\Gamma \backslash X(\infty)$ was canonically identified with $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$.

In [JM], a general procedure was given to compactify a complete (not necessarily simply connected) Riemannian manifold which satisfies certain conditions by adding the set of equivalence classes of EDM geodesics. It was then shown that these conditions were satisfied by $\Gamma \backslash X$, and hence the geodesic compactification $\Gamma \backslash X \cup \Gamma \backslash X(\infty)$ exists. Since $\Gamma \backslash X(\infty) = \Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$, the geodesic compactification is the same as the Tits compactification $\overline{\Gamma \backslash X}^T$.

Gromov compactification

In [BGS], Gromov introduced a compactification of any complete Riemannian manifold M using the distance functions. Let $C^0(M)$ be the space of continuous functions on M , and $\tilde{C}^0(M)$ be the quotient of $C^0(M)$ by the space of constant functions. Define a map

$$i_G : M \rightarrow \tilde{C}^0(M), \quad x \mapsto [d(x, \cdot)],$$

where $d(x, \cdot)$ is the distance from x , and $[d(x, \cdot)]$ its image in $\tilde{C}^0(M)$. It can be shown that the map i_G is an embedding, and the closure of $i_G(X)$ in $\tilde{C}^0(M)$ is compact and called the Gromov compactification and denoted by \overline{M}^G .

When M is a symmetric space X of noncompact type, it was shown in [BGS] that $\overline{X}^G \cong X \cup X(\infty)$. When $M = \Gamma \backslash X$, it was shown in [JM] that $\overline{\Gamma \backslash X}^G \cong \overline{\Gamma \backslash X}^T$ and hence

$$\overline{\Gamma \backslash X}^G = \Gamma \backslash X \cup \Gamma \backslash X(\infty).$$

Tangent cone at infinity

For any Riemannian manifold (M, g) , and a point $p \in M$, the tangent space $T_p M$ can be obtained as the Gromov-Hausdorff limit of $(M, \varepsilon g)$ with the fixed point p as $\varepsilon \rightarrow 0$. On the other hand, if the Gromov-Hausdorff limit of $(M, \frac{1}{\varepsilon} g)$ exists as $\varepsilon \rightarrow 0$, it called the tangent space of M at infinity, denoted by $T_{\infty} M$ (For general manifolds, we need to take ultralimits in order to get existence of the tangent cone. See §4 below about how the tangent cone of symmetric spaces leads to \mathbb{R} -trees and \mathbb{R} -Euclidean buildings). Clearly, it is independent of the choice of the point p and only depends on the quasi-isometry class of the metric g , and hence describes the geometry at infinity. It was shown in [JM] that $T_{\infty} \Gamma \backslash X$ exists and is equal to a cone $C(\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G}))$, a metric cone with a homothety section given by $\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G})$.

In [J3], it was shown that the Gromov-Hausdorff distance between $\Gamma \backslash X$ and $C(\Gamma \backslash \Delta_{\mathbb{Q}}(\mathbf{G}))$ is finite. This is closely related to a conjecture of Siegel which compares metrics of X and $\Gamma \backslash X$ on Siegel sets. See [J3] for details.

3.10 Cohomology of arithmetic groups

Given any discrete group Γ , the homology and cohomology groups of Γ are important homological invariants of Γ . For any $\mathbb{Z}\Gamma$ -module E , there are cohomology

and homology groups $H^i(\Gamma, E)$, $H_i(\Gamma, E)$. Usually, they are defined in terms of projective resolutions of E , $\mathbf{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow E$. The homology is defined by

$$H_i(\Gamma, E) = H_i(\mathbf{F} \otimes_{\Gamma} E), \quad (3.10.1)$$

where $\mathbf{F} \otimes_{\Gamma} E$ is the complex given by $F_i \otimes_{\Gamma} E = F_i \otimes E / \sim$, and the relation \sim is defined by $f \otimes m \sim \gamma f \otimes \gamma m$, for $f \in F_i, m \in E, \gamma \in \Gamma$. The cohomology is defined by

$$H^i(\Gamma, E) = H^i(\mathcal{H}om_{\Gamma}(F, E)). \quad (3.10.2)$$

See [Br3, Chap. III] for more details.

They can also be defined topologically in terms of the classifying space $B\Gamma$ of Γ . Recall that the classifying space $B\Gamma$ is a CW-complex which is characterized unique up to homotopy equivalence by the following conditions:

$$\pi_1(B\Gamma) = \Gamma, \quad \pi_i(B\Gamma) = \{1\} \quad i \geq 2.$$

The universal covering space $E\Gamma = \widetilde{B\Gamma}$ is characterized by the following conditions:

1. Γ acts properly and fixed point freely on $E\Gamma$.
2. $E\Gamma$ is contractible.

The projection $E\Gamma \rightarrow B\Gamma$ is a principal Γ -bundle over $B\Gamma$, and Γ -principal bundles over any topological space B correspond to maps $B \rightarrow B\Gamma$, and hence $B\Gamma$ is the classifying space for Γ -principal bundles.

Then for the trivial $\mathbb{Z}\Gamma$ -module \mathbb{Z} ,

$$H^i(\Gamma, \mathbb{Z}) = H^i(B\Gamma, \mathbb{Z}), \quad H_i(\Gamma, \mathbb{Z}) = H_i(B\Gamma, \mathbb{Z}).$$

For a general $\mathbb{Z}\Gamma$ -module E , there is a local system on $E\Gamma \otimes_{\Gamma} E$ over $B\Gamma$, still denoted by E , and

$$H^i(\Gamma, E) = H^i(B\Gamma, E), \quad H_i(\Gamma, E) = H_i(B\Gamma, E).$$

For more details, see [J7]. The cohomological dimension of Γ , denoted by $\text{cd } \Gamma$, is defined by

$$\text{cd } \Gamma = \sup\{i \in \mathbb{Z} \mid H^i(\Gamma, E) \neq 0 \text{ for some } \mathbb{Z}\Gamma\text{-module } E\}.$$

If there exists a finite dimensional $B\Gamma$ -space, (or strictly speaking, a model of $B\Gamma$ -space), then

$$\text{cd } \Gamma \leq \dim B\Gamma.$$

Another finiteness condition of a model of $B\Gamma$ -space is that it is a finite CW-complex. Such a $B\Gamma$ -space is called a finite $B\Gamma$ -space. Existence of a finite $B\Gamma$ -space implies that in every degree $H^i(\Gamma, \mathbb{Z})$ and $H_i(\Gamma, \mathbb{Z})$ are finitely generated.

From these discussions, it is clear that a good model of $B\Gamma$ is important for understanding cohomological properties of Γ , for example, in computing $H^i(\Gamma, \mathbb{Z})$ and $H_i(\Gamma, \mathbb{Z})$.

It is known that if Γ contains nontrivial torsion elements, then $\text{cd } \Gamma = +\infty$, and hence there does not exist any finite dimensional models of $B\Gamma$ -spaces (or $E\Gamma$ -spaces).

If Γ contains a finite index torsion-free subgroup Γ' , then $\text{cd } \Gamma'$ is independently of the choice of Γ' , and the common dimension $\text{cd } \Gamma'$ is called the virtual cohomological dimension of Γ , denoted by $\text{vcd } \Gamma$.

For groups Γ containing torsion elements, a more natural replacement for $E\Gamma$ is the universal space for proper actions of Γ , denoted by $\underline{E}\Gamma$, which is characterized by the following conditions:

1. Γ acts properly on $\underline{E}\Gamma$.
2. For every finite subgroup $F \subset \Gamma$, the set of fixed point $(\underline{E}\Gamma)^F$ is nonempty and contractible. In particular, when F is the trivial subgroup, $\underline{E}\Gamma$ is contractible.

Of course, when Γ is torsion-free, $\underline{E}\Gamma$ is reduced to $E\Gamma$. When $\underline{E}\Gamma$ is a Γ -CW complex and the quotient $\Gamma \backslash \underline{E}\Gamma$ is a finite CW-complex, then it is called a cofinite $\underline{E}\Gamma$ -space. The existence of a cofinite $\underline{E}\Gamma$ implies that Γ contains only finitely many conjugacy classes of finite subgroups.

One effective method to compute the cohomological dimension $\text{cd } \Gamma$ is to show that Γ is a duality group (or a generalized Poincaré duality group).

Recall that if $B\Gamma$ is given by a compact connected oriented manifold M of dimension n , then the Poincaré duality property for M implies that the cohomology and homology groups of Γ satisfy the following duality property: for any $i \geq 0$,

$$H^i(\Gamma, E) \cong H_{n-i}(\Gamma, E). \quad (3.10.3)$$

Motivated by this, we define that if the group Γ acts on \mathbb{Z} , i.e., \mathbb{Z} is a $\mathbb{Z}\Gamma$ -module, and satisfies the above condition in Equation (3.10.3) for all $\mathbb{Z}\Gamma$ -module E , then Γ is called a *Poincaré duality group* of dimension n . If the action of Γ on \mathbb{Z} is trivial, then $\mathbb{Z} \otimes E \cong E$, and Γ is called an *orientable* Poincaré duality group. Otherwise, Γ is a *non-orientable* Poincaré duality group.

More generally, a group Γ is called a *duality group* (or a *generalized Poincaré duality group*) of dimension n if there exists a $\mathbb{Z}\Gamma$ -module D such that for every $i \geq 0$ and every $\mathbb{Z}\Gamma$ -module E , there exists an isomorphism

$$H^i(\Gamma, E) \cong H_{n-i}(\Gamma, D \otimes E). \quad (3.10.4)$$

In this case, the module D is called the *dualizing module* of Γ . It is known that under the assumption that Γ is of type FP , if D is finitely generated abelian group, then D must be isomorphic to \mathbb{Z} which is regarded as a Γ -module, and hence Γ is a Poincaré duality group of dimension n .

A group Γ is called a *virtual Poincaré duality group* if there exists a subgroup of finite index Γ' which is a Poincaré duality group. The notion of *virtual duality groups* can be defined similarly.

Proposition 3.10.1. *If Γ is a duality group of dimension n , then Γ is of type FP, and the cohomological dimension of Γ is equal to n ; in particular, Γ is torsion-free. If Γ is a virtual duality group of dimension n , then the virtual cohomological dimension of Γ is equal to n .*

The existence of a good model of $B\Gamma$ is often the key to showing that Γ is a duality group. As mentioned before, if a group Γ admits a $B\Gamma$ -space given by a compact manifold M without boundary, then Γ is a Poincaré duality group.

If the *compactness* condition on $B\Gamma$ is dropped, or $B\Gamma$ is given by a compact manifold with *nonempty boundary*, then it is not obvious if Γ is a Poincaré duality group or even a duality group. For example, if Γ is a torsion-free non-uniform arithmetic subgroup of a semisimple linear algebraic group, such as a torsion-free subgroup of $SL(2, \mathbb{Z})$ of finite index, then there are $B\Gamma$ -spaces given by compact manifold with boundary but Γ is not a Poincaré duality group.

A useful algebraic criterion for duality is the following one.

Proposition 3.10.2. *Let Γ be a group of type FP. Then Γ is a duality group of dimension n if and only if one of the following equivalent conditions holds:*

1. $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$, and $H^n(\Gamma, \mathbb{Z}\Gamma)$ is a torsion-free abelian group.
2. for every abelian group E , the cohomology group $H^i(\Gamma, \mathbb{Z}\Gamma \otimes E) = 0$ for $i \neq n$.

If the above conditions are satisfied, then $H^n(\Gamma, \mathbb{Z}\Gamma)$ is the dualizing module D of Γ . As an abelian group, D is either isomorphic to \mathbb{Z} or infinitely generated.

On the other hand, the following more geometrical (or topological) result is very useful in studying the duality of Γ .

Proposition 3.10.3. *Suppose that a $B\Gamma$ -space is given by a finite CW-complex. Let $E\Gamma = \overline{B\Gamma}$ be the universal covering space of $B\Gamma$. Then*

$$H^i(\Gamma, \mathbb{Z}\Gamma) \cong H_c^i(E\Gamma, \mathbb{Z}).$$

In particular, if Γ is a Poincaré duality group of dimension n , then its dualizing module is equal to $H_c^n(E\Gamma, \mathbb{Z})$.

Suppose that a $B\Gamma$ -space is given by an n -dimensional smooth compact manifold with nonempty boundary. Then its universal covering space $E\Gamma$ is a contractible manifold with boundary. Fix an orientation of $E\Gamma$.

Proposition 3.10.4. *Assume that Γ admits a $B\Gamma$ -space given by an n -dimensional smooth compact manifold with nonempty boundary as above. If the boundary $\partial E\Gamma$ is homotopy equivalent to a bouquet of spheres S^{r-1} of dimension $r-1$ (i.e., a wedge product of S^{r-1}), then Γ is a duality group of dimension $n-r$, and the dualizing module is equal to the reduced homology of $\partial E\Gamma$ in degree $r-1$. Furthermore, if the bouquet contains at least one but only finitely many spheres S^{r-1} , then it must contain exactly one sphere, and Γ is a Poincaré duality group in this case; otherwise, the bouquet contains infinitely many spheres and Γ is not a Poincaré duality group.*

Proof. By the Poincaré duality for manifolds with boundary,

$$H_c^i(E\Gamma, \mathbb{Z}) \cong H_{n-i}(E\Gamma, \partial E\Gamma, \mathbb{Z}).$$

By the long exact sequence for the pair $E\Gamma, \partial E\Gamma$, we obtain that

$$H_c^i(E\Gamma, \mathbb{Z}) \cong \tilde{H}_{n-i-1}(\partial E\Gamma, \mathbb{Z}).$$

Then the assumption on the homotopy type of $E\Gamma$ implies that $H_c^i(E\Gamma, \mathbb{Z})$ is equal to 0 for $i \neq n-r$ and is free for $n-r$, and hence Γ is a duality group of dimension $n-r$ by Proposition 3.10.2.

In the above proposition, if $B\Gamma$ is a closed manifold, the boundary $\partial E\Gamma$ is empty and hence of dimension -1 , i.e., $r = 0$. This implies that Γ is a duality group of dimension n .

Remark 3.10.5. *Note that every smooth compact manifold with corners is homeomorphic to a smooth manifold with boundary by smoothing out the corners, and we can also assume the $B\Gamma$ -space in the above proposition is a compact manifold with corners.*

A related result on cohomological dimension is the following [Br3, Proposition 8.1, p. 210].

Proposition 3.10.6. *Suppose that a $B\Gamma$ -space is a compact manifold with non-empty boundary. Then $\text{cd } \Gamma \leq \dim B\Gamma - 1$.*

For general discrete groups Γ , it is usually difficult to compute its virtual cohomological dimension and to decide whether it is a duality group. When Γ is an arithmetic subgroup of a linear algebraic group \mathbf{G} over \mathbf{G} , then all these questions can be answered by making use of the Borel-Serre compactification of locally symmetric spaces and the Solomon-Tits theorem.

Note that a manifold with corners can be deformed to a manifold with boundary by smoothing out corners. Therefore, by Proposition 3.8.2, \overline{X}^{BS} is a model of $E\Gamma$ satisfies all the conditions of Proposition 3.10.4. Since every arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ contains a finite index torsion-free subgroup, we have the following important result from [BoS2].

Proposition 3.10.7. *Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of a semisimple linear algebraic group \mathbf{G} as above. Then Γ is a virtual duality group of dimension $n-r$, where $n = \dim X = G/K$, and r is the \mathbb{Q} -rank of \mathbf{G} . The dualizing module is equal to the reduced homology of the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ in degree $r-1$, which is usually called the Steinberg module. In particular, the virtual cohomological dimension of Γ is equal to $n-r$.*

See [AbB, Chapter 13] [Ro2] for more details and references.

3.11 Vanishing of simplicial volume of high rank locally symmetric spaces

In this subsection, we explain another application of the Borel-Serre compactification and the Solomon-Tits Theorem to vanishing of simplicial volumes of locally symmetric spaces of \mathbb{Q} -rank at least 3.

The simplicial volume is an important homotopy invariant of manifolds introduced by Gromov [Gr2]. It is defined as follows. Suppose that M is a connected oriented compact manifold of dimension n , let $[M]$ be the fundamental class in $H_n(M, \mathbb{R})$, or rather the image of the fundamental class in $H_n(M, \mathbb{Z})$ under the natural map $H_n(M, \mathbb{Z}) \rightarrow H_n(M, \mathbb{R})$. For each n -chain $c = \sum_{\sigma} a_{\sigma} \sigma$, where σ are n -singular simplices, define the simplicial ℓ^1 -norm

$$\|c\|_1 = \sum_{\sigma} |a_{\sigma}|.$$

Then the *simplicial volume* of M , denoted by $\|M\|$, is defined by

$$\|M\| = \inf\{\|c\|_1 \mid c \text{ is an } n\text{-chain in the fundamental class } [M]\}.$$

If M is a connected non-orientable manifold, then let \tilde{M} be its double cover and define $\|M\| = \frac{1}{2}\|\tilde{M}\|$. If M is an oriented orbifold, then it has a fundamental class and hence the usual notion of simplicial volume. If M admits a finite smooth cover N , then it also has the *orbifold simplicial volume* $\|M\|_{orb} = \|N\|/d$, where d is the degree of the covering $N \rightarrow M$. It is known that

$$\|M\|_{orb} \geq \|M\|,$$

and the strict inequality can occur.

Assume that M is a connected orientable noncompact manifold of dimension n . Let $H_*^{\text{lf}}(M, \mathbb{R})$ be the locally finite homology group of M . Let $[M]^{\text{lf}}$ be the fundamental class in $H_*^{\text{lf}}(M, \mathbb{R})$. For any locally finite n -chain $c = \sum_{\sigma} a_{\sigma} \sigma$, define the simplicial norm $\|c\|_1 = \sum_{\sigma} |a_{\sigma}|$ as above. Recall that a chain is locally finite means that every compact subset of M only meets the images of finitely many singular simplices σ in the chain. Hence, $\|c\|_1$ could be equal to infinity.

The simplicial volume $\|M\|$ of a noncompact manifold M is defined by

$$\|M\| = \inf\{\|c\|_1 \mid c \text{ is a locally finite } n\text{-chain in the fundamental class } [M]^{\text{lf}}\}.$$

One of the motivations of Gromov to introduce the simplicial volume is to give a lower bound on the minimal volume of a manifold M . Consider all complete Riemannian metrics g on M whose sectional curvature $K(g)$ satisfies the bound $|K(g)| \leq 1$ at all points. Let $\text{Vol}(M, g)$ denote the volume of M with respect to the metric g . Then the minimal volume of M is defined by

$$\text{Min-Vol}(M) = \inf\{\text{Vol}(M, g) \mid \text{all complete metrics } g, |K(g)| \leq 1\}.$$

Another major application of the simplicial volume is a different proof by Gromov of Mostow strong rigidity for compact hyperbolic spaces of dimension at least 3 as mentioned above. See [BeP] [Rat] for detailed discussions.

A basic result in [Gr2, §0.5] states that there exists a universal constant C_n only depending on the dimension n such that

$$\text{Min-Vol}(M) \geq C_n \|M\|.$$

Therefore, a natural problem is to understand when the simplicial volume $\|M\|$ is equal to zero.

It is known that if a compact manifold M admits a self-map of degree greater than or equal to 2, then its simplicial volume $\|M\| = 0$. If a noncompact manifold M admits a proper self-map of degree greater than or equal to 2, then $\|M\| = 0$ or $\|M\| = +\infty$. As a consequence, the simplicial volumes of spheres and tori are equal to zero. It is also known that the simplicial volume of \mathbb{R}^n is equal to 0.

For spaces related to locally symmetric spaces, the following results on simplicial volume are known:

Proposition 3.11.1. *If M is a complete hyperbolic manifold of finite volume, then $\|M\| > 0$. More generally, if M admits a complete metric such that its sectional curvature K is bounded between two negative constants uniformly, then $\|M\| > 0$.*

This is a result due to Thurston [Gr2, §0.3].

Proposition 3.11.2. *If $M = \Gamma \backslash X$ is a compact locally symmetric space of noncompact type, then $\|M\| > 0$.*

This was conjectured by Gromov [Gr2, p. 11] and proved in [LaS] and [BuK].

Proposition 3.11.3. *If $M = \Gamma \backslash X$ is an arithmetic locally symmetric space whose \mathbb{Q} -rank, denoted by $r_{\mathbb{Q}}(\mathbf{G})$, is greater than or equal to 3, then $\|M\| = 0$.*

The vanishing result was proved in [LoS2]. In the proof, a vanishing criterion [Gr2, p. 58] was applied. In order to apply this, a suitable covering of $\Gamma \backslash X$ with multiplicity at most $\dim \Gamma \backslash X$ is needed. For this purpose, the conditions that the map $\pi_1^{\infty}(\Gamma \backslash X) = \pi_1(\overline{\Gamma \backslash X}^{BS}) \rightarrow \pi_1(\Gamma \backslash X) = \pi_1(\overline{\Gamma \backslash X}^{BS})$ is injective and that the virtual cohomological dimension of Γ is at most $\dim \Gamma \backslash X - 2$ are needed. Since the virtual cohomological dimension of Γ is equal to $\dim \Gamma \backslash X - r_{\mathbb{Q}}(\mathbf{G})$, the condition that $r_{\mathbb{Q}}(\mathbf{G}) \geq 3$ is more than enough. The assumption that $r_{\mathbb{Q}}(\mathbf{G}) \geq 3$ is needed to show that the boundary $\partial_{\mathbb{Q}} \overline{X}^{BS}$, which is homotopy equivalent to a bouquet of spheres $S^{r_{\mathbb{Q}}(\mathbf{G})-1}$ by Proposition 3.8.2, is simply connected. From this brief outline, it is clear that the Solomon-Tits Theorem for the spherical building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is used crucially.

If $r_{\mathbb{Q}}(\mathbf{G}) = 1$, this vanishing result does not hold in general. For example, it was proved in [LoS1] that if M is a Hilbert modular variety, then $\|M\| > 0$. Note that Hilbert modular varieties are important examples of locally symmetric spaces of \mathbb{Q} -rank 1. If the rank of X is equal to 1 and $\Gamma \backslash X$ is noncompact, which implies that $r_{\mathbb{Q}}(\mathbf{G}) = 1$, then the sectional curvature of $\Gamma \backslash X$ is bounded by two negative constants and hence the simplicial volume of $\Gamma \backslash X$ is positive [Gr2, §0.3].

3.12 Generalizations of buildings: curve complexes and applications

As discussed in the previous subsection, spherical buildings of semisimple algebraic groups defined over \mathbb{Q} give natural parameter spaces of the boundary components of the Borel-Serre partial compactifications of symmetric spaces, and the topology of the buildings, i.e., the Solomon-Tits theorem, has played an important role in understanding the topology of compactifications of locally symmetric spaces and cohomological properties of arithmetic groups.

Motivated by the Borel-Serre compactification of locally symmetric spaces, Harvey introduced curve complexes of surfaces, which are infinite simplicial complexes and parametrize boundary components of an analogue of the Borel-Serre partial compactification of the Teichmüller spaces. The curve complexes have turned out to have important applications to cohomology properties of mapping class groups in a similar way as spherical Tits buildings are used for symmetric spaces and arithmetic groups.

We mention some definitions, basic properties and applications. For detailed discussions and references of definitions and results stated here, see [J8]. For simplicity, we only deal with compact surfaces. Let S_g be a compact oriented surface of genus g ,

The curve complex $\mathcal{C}(S_g)$ is a simplicial complex whose vertices correspond to homotopy equivalence classes of simple closed curves. Homotopy classes of simple closed curves $[c_1], \dots, [c_{k+1}]$ form the vertices of a k -simplex if and only if they admit disjoint representatives. By assigning each edge length 1, $\mathcal{C}(S_g)$ becomes a metric space.

Proposition 3.12.1. *The curve complex $\mathcal{C}(S_g)$ has infinite diameter and is a Gromov hyperbolic space.*

The analogue of the Solomon-Tits Theorem also holds.

Proposition 3.12.2. *The curve complex $\mathcal{C}(S_g)$ is homotopy equivalent to a bouquet of infinitely many spheres of dimension $2g - 2$.*

Let \mathcal{T}_g be the Teichmüller space of marked complex structures on S_g . It is a complex manifold and diffeomorphic to \mathbb{R}^{6g-3} . Let $Mod_g = \text{Diff}^+(S_g)/\text{Diff}^0(S_g)$ be the mapping class group of S_g , where $\text{Diff}^+(S_g)$ is the group of orientation preserving diffeomorphisms of S_g , and $\text{Diff}^0(S_g)$ is the identity component of $\text{Diff}^+(S_g)$. Then Mod_g acts holomorphically and properly on \mathcal{T}_g , and the quotient $Mod_g \backslash \mathcal{T}_g$ is the moduli space of Riemann surfaces of genus g and is noncompact.

Instead of partially compactifying \mathcal{T}_g to get a compact quotient by Mod_g , we can consider the thick part $\mathcal{T}_g(\varepsilon)$. Assume that $g \geq 2$. Then every compact Riemann surface of genus g admits a unique hyperbolic metric conformal to the complex structure. For small $\varepsilon > 0$, the thick part $\mathcal{T}_g(\varepsilon)$ consists of marked Riemann surfaces that do not contain any simple closed geodesic of length less than ε .

It is clear that Mod_g acts on $\mathcal{T}_g(\varepsilon)$. The following results can be proved.

Proposition 3.12.3. *When ε is sufficiently small, the thick part $\mathcal{T}_g(\varepsilon)$ is a manifold with corners and is contractible, and the quotient $Mod_g \backslash \mathcal{T}_g(\varepsilon)$ is compact. The boundary faces (or components) of $\mathcal{T}_g(\varepsilon)$ are contractible and parametrized by simplices of the curve complex $\mathcal{C}(S_g)$, and hence the boundary $\partial\mathcal{T}_g(\varepsilon)$ is homotopy equivalent to a bouquet of spheres of dimension $2g - 2$. $\mathcal{T}_g(\varepsilon)$ is a cofinite model of the universal space $\underline{E}Mod_g$ for proper actions of Mod_g .*

It is known that the mapping class group Mod_g is virtually torsion-free, i.e., admits torsion-free finite index subgroups. As a consequence of the above results, by following the arguments in [BoS2] (see Proposition 3.10.7), we have the following result of Harer.

Proposition 3.12.4. *Mod_g is a virtual duality group of dimension $4g - 5$, and the dualizing module is equal to the reduced homology of $\mathcal{C}(S_g)$ in degree $2g - 2$.*

4 Euclidean buildings

In this section, we discuss Euclidean reflection groups and Euclidean buildings, and construct the Bruhat-Tits buildings of semisimple simply connected algebraic groups defined over a field with discrete valuation. The key point is that instead of parabolic subgroups, we use parahoric subgroups to parametrize simplexes of the buildings. Then we discuss choices of BN -pairs and buildings for non-simply connected algebraic groups and a concrete realization of the Bruhat-Tits building of the group $SL(n)$.

For applications to S -arithmetic subgroups, we need to compactify Euclidean buildings by spherical Tits buildings. We also discuss some Satake type compactifications of Euclidean buildings.

4.1 Definitions and basic properties

Let V be a Euclidean space. An affine reflection group W on V is a group of affine isometries generated by reflections with respect to affine hyperplanes such that the set \mathcal{H} of affine hyperplanes fixed by reflections in W is locally finite. Clearly, a finite reflection group is an affine reflection group.

The linear parts of the affine transformations in W define a finite (linear) reflection group \bar{W} . W is called essential if \bar{W} is essential.

The hyperplanes in \mathcal{H} divide V into chambers, and W acts simply transitively on the set of chambers. Let C be a chamber. Then W is generated by the reflections of the walls of C .

Proposition 4.1.1. *Assume that W is essential and irreducible. Then*

1. *either W is finite and has a fixed point, and hence becomes a finite reflection group when the point becomes the origin. In this case, C is a simplicial cone;*
2. *or W is infinite and C is a simplex.*

See [Br1, Chap VI., §1] for a proof of this proposition. An essential infinite affine reflection group is called a Euclidean reflection group. The chambers and their faces form a simplicial complex, which is called a *Euclidean Coxeter complex*. A typical example of a Euclidean reflection group is the affine Weyl group of a simple Lie algebra over \mathbb{C} .

Definition 4.1.2. *A chamber complex Δ is called a Euclidean building if it satisfies all the conditions in Definition 2.1.4 except that all the apartments are Euclidean Coxeter complexes.*

Since the underlying space of each Euclidean Coxeter complex is a Euclidean space, it has a metric. Fix a Euclidean metric on every apartment such that all apartments are isometric. Then the compatibility conditions on apartments shows that these metrics can be glued into a metric on Δ which is a geodesic space, i.e., the distance between any two points is realized by a geodesic connecting them [BH1] (see also [Br1, Chap. VI, §3]).

Proposition 4.1.3. *Any Euclidean building Δ as a metric space is a CAT(0)-space, and hence has nonpositive curvature and is contractible. In particular, it is simply connected.*

Recall that a CAT(0)-space M is a length space such that every triangle in M is thinner than a corresponding triangle in \mathbb{R}^2 of the same side lengths [BH]. When Δ is an Euclidean building of rank 1, i.e., a tree, then it is clear that it is a CAT(0)-space.

This proposition has important application to understanding compact subgroups of semisimple p -adic groups.

4.2 Semisimple p -adic groups and Euclidean buildings

Next we consider some examples of Euclidean buildings constructed from linear semisimple algebraic groups defined over local fields.

Let \mathbf{G} be a linear connected semisimple algebraic group defined over a field complete with respect to a discrete valuation. An important class of such fields F consists of non-archimedean local fields F , i.e., locally compact fields complete with respect to a discrete valuation, which is equivalent to the condition that the residue field f of F is finite. For example, we can take $F = \mathbb{Q}_p$, where p is a finite prime, and the residue field is $f = \mathbb{F}_p$. We also assume that \mathbf{G} is absolutely almost simple, i.e., over any field extension of F , \mathbf{G} has no proper normal subgroup of strictly positive dimension.

In the first part, we assume that \mathbf{G} is *simply connected*, i.e., there is no non-trivial isogeny $\mathbf{G}' \rightarrow \mathbf{G}$ over K . (Recall that an isogeny $\mathbf{G}' \rightarrow \mathbf{G}$ is a surjective homomorphism f which has finite kernel, and which is also a morphism in the sense of algebraic varieties.) Examples include $\mathbf{G} = SL(n), Sp(n)$. In the second part, we consider the non-simply connected case. Then we conclude this subsection with the example of the Bruhat-Tits building of $SL(n)$.

Simply connected semisimple algebraic groups

In §2.2, for any linear semisimple algebraic group \mathbf{G} defined over any field F , we can associate a spherical Tits building $\Delta(\mathbf{G})$, whose simplexes correspond to parabolic subgroups of \mathbf{G} which are defined over F . When the parabolic subgroups are replaced by parahoric subgroups of $\mathbf{G}(F)$, we obtain a Euclidean building $\Delta^{BT}(\mathbf{G})$, called the Bruhat-Tits building of \mathbf{G} .

As recalled in §2.2, parabolic subgroups \mathbf{P} of \mathbf{G} can be defined simply as subgroups such that \mathbf{G}/\mathbf{P} is a projective variety. On the other hand, there is no such simple characterization of parahoric subgroups of $\mathbf{G}(F)$ for general reductive group \mathbf{G} (see [BT2, Definition 5.2.6] and also [BT1, p.31]).

Fortunately, under our assumption that \mathbf{G} is *simply connected* and *absolutely almost simple* which means G does not contain any nontrivial connected normal subgroups over the algebraic closure of F , there is a direct definition of parahoric subgroups. A minimal parahoric subgroup of $\mathbf{G}(F)$ is called an Iwahori subgroup and is equal to the normalizer of any maximal pro- p -subgroup of $\mathbf{G}(F)$, where p is the characteristic of the residue field f , and a pro- p -subgroup is the projective limit of a sequence of p -subgroups. Such minimal parahoric subgroups are conjugate under $\mathbf{G}(F)$. Then any proper subgroup of $\mathbf{G}(F)$ containing an Iwahori subgroup is called a parahoric subgroup of $\mathbf{G}(F)$.

When $\mathbf{G} = SL(n)$, examples of Iwahori and parahoric subgroups are described explicitly in [Br1, Chap. V, §8] (see below).

The relations between the parahoric subgroups are similar to those of parabolic subgroups.

Proposition 4.2.1. *Under the above assumptions on \mathbf{G} , all minimal parahoric subgroups of $\mathbf{G}(F)$ are conjugate. Fix any minimal parahoric subgroup B . Let r be the rank of \mathbf{G} over F . Then there are exactly $r + 1$ maximal parahoric subgroups P_1, \dots, P_{r+1} which contain B , and $P_{i_1} \cap \dots \cap P_{i_k}$ are exactly all the parahoric subgroups which contain B when $\{i_1, \dots, i_k\}$ runs over non-empty subsets of $\{1, \dots, r + 1\}$. In particular, any parahoric subgroup is conjugate to such an intersection.*

Parahoric subgroups are basically related to bounded subgroups and compact open subgroups of $\mathbf{G}(F)$.

Definition 4.2.2. *A subgroup $K \subset \mathbf{G}(F)$ is called bounded if it satisfies one of the following equivalent conditions:*

1. *There exists a faithful finite dimensional rational representation $\rho : \mathbf{G}(F) \rightarrow GL(n, F)$ defined over F such that the coordinates of the elements $\rho(g)$, $g \in K$, are uniformly bounded with respect to the natural metric of F .*
2. *For any finite dimensional rational representation $\rho : \mathbf{G}(F) \rightarrow GL(n, F)$, the coordinates of the elements $\rho(g)$, $g \in K$, are uniformly bounded with respect to the natural metric of F .*
3. *For any finite dimensional rational representation $\rho : \mathbf{G}(F) \rightarrow GL(n, F)$, there exists a lattice Λ in F^n such that $\rho(g)\Lambda = \Lambda$, $g \in K$.*

By (3), any bounded subgroup is contained in a compact open subgroup of $\mathbf{G}(F)$. When \mathbf{G} is simply connected, any maximal open compact subgroup of $\mathbf{G}(F)$ is a maximal parahoric subgroup, and the converse is also true.

Remark 4.2.3. To emphasize the similarities between the parabolic subgroups and parahoric subgroups, we point out that when \mathbf{G} is split over F , then the minimal parahoric subgroups, i.e., the Iwahori subgroups, are lifted from the Borel subgroups of the reduction modulo the prime ideal of some integral model of \mathbf{G} (see [IM] [Iw] and the example of $\mathbf{G} = SL(n)$ in [Br1, p. 130]). In this split case, the minimal parabolic subgroups of \mathbf{G} are given by the Borel subgroups, i.e., the maximal connected solvable subgroups.

The Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is defined to be the simplicial complex whose simplexes correspond to parahoric subgroups such that

1. The maximal parahoric subgroups correspond to vertices.
2. The inclusion relation is the opposite of the inclusion relation of the parahoric subgroups. In particular, for any parahoric subgroup P , the vertices of its simplex correspond to the maximal parahoric subgroups which contain P .

The apartments in $\Delta^{BT}(\mathbf{G})$ are also similarly described as in the case of the spherical Tits building $\Delta(\mathbf{G})$. Let \mathcal{O}_F be the ring of valuation. Then for any maximal F -split torus \mathbf{T} of \mathbf{G} , the set of parahoric subgroups which contain $\mathbf{T}(\mathcal{O}_F)$ form an apartment in $\Delta^{BT}(\mathbf{G})$ and is isomorphic to a Euclidean Coxeter complex.

As pointed out in §2.3, buildings can also be described in terms of BN -pairs. In this case, B is a minimal parahoric subgroup, and \mathbf{T} is a maximal F -split torus such that $\mathbf{T}(F)$ is contained in B . Let N be the normalizer of $\mathbf{T}(F)$ in $\mathbf{G}(F)$. Then B and N form a BN -pair, i.e., they satisfy the axioms in Definition 2.3.1. The intersection $B \cap N$ is a maximal compact subgroup of the centralizer of $\mathbf{T}(F)$ in $\mathbf{G}(F)$, and $W = N/(B \cap N)$ is a Euclidean Coxeter group whose Coxeter complex is isomorphic to the apartments in $\Delta^{BT}(\mathbf{G})$. In fact, W is the extension of a finite Coxeter group by a free abelian group of rank equal to the F -rank of \mathbf{G} . Such a Weyl group is called an *affine Weyl group*.

Since $\mathbf{G}(F)$ acts on the set of parahoric subgroups by conjugation, it acts simplicially on $\Delta^{BT}(\mathbf{G})$ such that the stabilizers of the vertices are the maximal parahoric subgroups.

Remark 4.2.4. When \mathbf{G} is absolutely almost simple and simply connected over $F = \mathbb{Q}_p$ or more general local fields, the existence of BN -pairs with affine Weyl groups in $\mathbf{G}(F)$ is proved in [Hij]. Such BN -pairs with affine Weyl groups are unique up to conjugation. On the other hand, for a general reductive linear group \mathbf{G} defined over a local field, the Bruhat-Tits buildings are constructed first using valued root data, and then parahoric subgroups are defined in terms of stabilizers of facets in Bruhat-Tits buildings, different from the procedure described above.

Remark 4.2.5. The basic idea in the approach using valued root data is to start with the structure of an apartment and to use the group action to obtain other apartments and hence the buildings. Roughly, let A be an apartment. Then the

Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is defined as a quotient of $G \times A$. Such an idea of passing from an apartment to the whole buildings was also used in constructing compactifications of Bruhat-Tits buildings in [Lan2]. It is also needed to construct non-discrete \mathbb{R} -Euclidean buildings discussed in §4 below.

A related idea of passing from a totally geodesic maximal flat subspace of a symmetric space X of noncompact type to the symmetric space has been used to understand compactifications of symmetric spaces in [Os] and [GJT, Chap. III].

Remark 4.2.6. For a semisimple algebraic group \mathbf{G} as in the above Remark, besides the BN -pair which gives the Bruhat-Tits building, there is also a BN -pair in §2.3 which leads to the spherical Tits building corresponding to the parabolic subgroups. The first BN -pair depends on the structure of the local fields, while the second depends on the structure of the algebraic group \mathbf{G} .

Non-simply connected semisimple groups

Let \mathbf{G} be a connected, non-simply connected linear semisimple algebraic group defined over a non-archimedean local field F . Let $\tilde{\mathbf{G}}$ be its universal covering. Then $\tilde{\mathbf{G}}$ is a simply connected semisimple linear algebraic group defined over F , and hence its Bruhat-Tits building of $\Delta^{BT}(\tilde{\mathbf{G}})$ is defined as above. Define the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ to be equal to $\Delta^{BT}(\tilde{\mathbf{G}})$. Let $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ be the canonical central isogeny. The group $\tilde{\mathbf{G}}(F)$ acts on the set of its parahoric subgroups via conjugation. Since the center of $\tilde{\mathbf{G}}(F)$ clearly acts trivially, this gives an action of $\mathbf{G}(F)$ on the set of parahoric subgroups of $\tilde{\mathbf{G}}(F)$ and hence on the building $\Delta^{BT}(\tilde{\mathbf{G}})$ (see [BoS1, §4.8]).

Example: Bruhat-Tits building of $SL(n)$.

For the sake of explicitness, we describe the BN -pair in the case $\mathbf{G} = SL(n)$ over \mathbb{Q}_p , where p is a finite prime [Br1, Chap. V, §8]. Let $SL(n, \mathbb{F}_p) \rightarrow SL(n, \mathbb{F}_p)$ be the natural projection. Take B to be the inverse image of the upper triangular subgroup, and N to be the monomial subgroup of $SL(n, \mathbb{Q}_p)$, i.e., the subgroup consisting of all matrices with exactly one non-zero element in every row and every column.

The simplexes and apartments in $\Delta^{BT}(SL(n))$ can also be described explicitly in terms of lattices (see [Ro1, pp. 31-32] [Br1, pp. 132-137]). Briefly, two lattices Λ and Λ' in F^n are equivalent if there exists an element $\lambda \in F^\times$ such that $\Lambda = \lambda\Lambda'$. The stabilizers of equivalent lattices are obviously the same, and they are the maximal parahoric subgroups. Hence, the equivalence classes of lattices correspond to vertices of $\Delta^{BT}(SL(n))$. Two equivalence classes $[\Lambda]$ and $[\Lambda']$ form the vertices of a simplex of dimension 1 if and only if there exist representatives Λ and Λ' such that

$$\pi\Lambda \supset \Lambda' \supset \Lambda,$$

where π is a uniformizer of the valuation ring of integer of the local field F . The vertices $[\Lambda_1], \dots, [\Lambda_m]$ form the vertices of a m -simplex if there are representatives $\Lambda_1, \dots, \Lambda_m$ such that

$$\Lambda_1 \supset \dots \supset \Lambda_m \supset \pi\Lambda_1.$$

If a lattice Λ has a basis v_1, \dots, v_n , denote the equivalence class of Λ by $[v_1, \dots, v_n]$. Let e_1, \dots, e_n be a basis of F^n . Then the vertices $[e_1, \dots, e_i, \pi e_{i+1}, \dots, \pi e_n]$, $i = 1, \dots, n$, form the vertices of a chamber in $\Delta^{BT}(SL(n))$, a fundamental domain of the Euclidean Coxeter group acting on the Euclidean Coxeter complex.

The equivalence classes $[\pi^{i_1} e_1, \dots, \pi^{i_n} e_n]$ form the vertices of an apartment containing the above chamber, and any apartments are obtained from this apartment under the action of $SL(n, F)$.

Remark 4.2.7. See [BT3] [BT4] for concrete realization of the building in terms of norms for classical groups and [GY1] [GY2] for exceptional groups. See also [AbN] [Yu1] for other concrete realizations. See also [Yu2] for a survey of the Bruhat-Tits theory.

Remark 4.2.8. In this case, there are $n+1$ $SL(n, F)$ -conjugacy classes of maximal parahoric subgroups of $SL(n, F)$, which are also maximal compact subgroups of $SL(n, F)$. On the other hand, it is well-known that there is only one $SL(n, \mathbb{R})$ -conjugacy class of maximal compact subgroups in $SL(n, \mathbb{R})$. To help explain this difference, we note that if $K \subset SL(n, F)$ is a maximal compact subgroup, then for any $g \in GL(n, F)$, the conjugate gKg^{-1} is also a maximal subgroup in $SL(n, F)$. But this conjugation can not always be achieved by elements in $SL(n, F)$. (This problem will not arise for $SL(n, \mathbb{R})$ since we can always assume $\det g > 0$ and use $(\det g)^{-1/n} g \in SL(n, \mathbb{R})$ to get the same conjugated subgroup). In fact, $GL(n, F)$ acts transitively on the set of lattices in F^n , and hence all the maximal compact subgroups of $SL(n, F)$ are conjugate under $GL(n, F)$.

When the F -rank of \mathbf{G} is equal to 1, $\Delta^{BT}(\mathbf{G})$ is a tree. For a thorough discussion of trees and their applications to combinatorial group theory, see [Se1].

4.3 Compactification of Euclidean buildings by spherical buildings

For a connected linear semisimple algebraic group \mathbf{G} defined over a non-archimedean local field F , there are two buildings associated with \mathbf{G} : the spherical Tits building $\Delta(\mathbf{G})$ and the Bruhat-Tits Euclidean building $\Delta^{BT}(\mathbf{G})$. These two buildings are closely related. In fact, the former with a suitable topology can be used to compactify the latter. This compactification is similar to the geodesic compactification $X \cup X(\infty)$ of a symmetric space X of noncompact type in §3.1, and is very important for applications to the cohomology of S -arithmetic groups in [BoS1].

The construction was given in [BoS1] and also reviewed in [Br1] and [Ro3]. Here we recall the compactification, in particular its topology in a slightly different way, i.e., we view the Euclidean building as a CAT(0)-space [BH] [Pap].

As recalled earlier, a CAT(0)-space is a geodesic metric space whose triangles are thinner than the corresponding triangles of the same side lengths in \mathbb{R}^2 . A complete simply connected manifold of nonpositive curvature, i.e., a Hadamard manifold, is a CAT(0)-space, and CAT(0)-spaces are generalizations of Hadamard manifolds to the category of metric spaces.

Let (Δ, d) be a CAT(0)-space. A ray in Δ is an isometric embedding $\gamma : [0, +\infty) \rightarrow \Delta$. Two rays γ_1, γ_2 are called equivalent if

$$\limsup_{t \rightarrow +\infty} d(\gamma_1(t), \gamma_2(t)) < +\infty.$$

Let $\Delta(\infty)$ be the set of equivalence classes of rays in Δ .

Define a conic topology on $\Delta \cup \Delta(\infty)$ as follows: Let $o \in \Delta$ be any basepoint. It is known that every equivalence class $[\gamma]$ of rays contains a unique ray γ which starts from o , i.e., $\gamma(0) = o$. Define $\gamma(+\infty)$ to be the equivalence class $[\gamma]$. Then we get an extended ray

$$\gamma : [0, +\infty] \rightarrow \Delta \cup \Delta(\infty).$$

For any ray γ with $\gamma(0) = o$ and $r > 0$ and $\varepsilon > 0$, define a neighborhood of $\gamma(+\infty)$ by

$$U(\gamma(+\infty), r, \varepsilon) = \{\gamma'(t) \mid \gamma' \text{ is a ray, } \gamma'(0) = o, d(\gamma'(r), \gamma(r)) < \varepsilon, t \in (r, +\infty]\}.$$

When r runs over a sequence of numbers going to infinity, and ε over a sequence going to 0, $U(\gamma(+\infty), r, \varepsilon)$ forms a neighborhood basis of $\gamma(+\infty)$ in $\Delta \cup \Delta(\infty)$ (see [BH, pp. 263-264]).

Remark 4.3.1. Let $S(o, r) = \{x \in \Delta \mid d(x, o) = r\}$ be the sphere of radius r with center o . Then $\gamma(r), \gamma'(r) \in S(o, r)$. For a Hadamard manifold, the spheres $S(o, r)$ are all diffeomorphic for different values of r . This implies that in this case, we could only use one fixed r to get a neighborhood basis of $\gamma(+\infty)$. On the other hand, for a general CAT(0)-space, r needs to run over a sequence of numbers going to infinity. This can be seen clearly when Δ is an infinite tree.

In this topology of $\Delta \cup \Delta(\infty)$, an unbounded sequence x_j converges to a boundary point $\gamma(+\infty)$ if and only if the unique ray $\gamma_j(t)$ starting from o and passes through x_j converges to $\gamma(t)$ uniformly for t in compact subsets.

Proposition 4.3.2. *If Δ is complete, the space $\Delta \cup \Delta(\infty)$ is contractible.*

Proof. Let $o \in \Delta$ be a basepoint as above. Then each ray γ issuing from o extends continuously to an extended ray $\gamma : [0, +\infty] \rightarrow \Delta \cup \Delta(\infty)$. When γ runs over all the rays starting from o , the extended ray sweeps out $\Delta \cup \Delta(\infty)$. Now contract $\Delta \cup \Delta(\infty)$ along such each extended rays to the basepoint o using a fixed deformation retract of $[0, +\infty)$ to 0. It follows easily from the definition of the conic topology that they define a deformation retraction of $\Delta \cup \Delta(\infty)$ to the basepoint o .

Proposition 4.3.3. *If a CAT(0)-space Δ is complete and locally compact, then $\Delta \cup \Delta(\infty)$ is compact, and $\Delta \cup \Delta(\infty)$ is called the geodesic compactification.*

Proof. Since Δ is a geodesic space, the local compactness and completeness imply that it is a proper metric space [Ha, p. 84]. It suffices to show that any unbounded sequence x_j in Δ has a convergent subsequence. Let γ_j be the unique ray from o which passes through x_j . The properness of Δ implies that for any

$r > 0$, the closed ball $B(o, r)$ of radius r and center o is compact, which in turn implies that there is a subsequence of $\gamma_j(t)$ which converges to a ray $\gamma(t)$ uniformly for t in compact subsets. Then the corresponding subsequence of x_j converges to $\gamma(+\infty)$.

If Δ is a Euclidean building, then $\Delta(\infty)$ is the underlying space of a spherical Tits building. The details are given in [BoS1] (see also [Br1] [Ro3]). We comment briefly on the construction. First we define the notion of apartments of Δ . Let Σ be an apartment in Δ and identify it with \mathbb{R}^r . Let W the Euclidean Coxeter group, and \bar{W} be its linear part. Then \bar{W} is a finite Coxeter group and divides \mathbb{R}^n into finite many simplicial cones and their faces. (In fact, they are the chambers and chamber faces of the finite Coxeter complex associated with \bar{W}). The corresponding subsets in Σ are called the sectors and sector faces in Δ .

For each sector C of Δ , let \bar{C} be the closure of C in $\Delta \cup \Delta(\infty)$. Define

$$\bar{C}(\infty) = \bar{C} \cap X(\infty).$$

Then $\bar{C}(\infty)$ is a closed simplex. Denote its interior by $C(\infty)$, called the simplex at infinity associated with C . For any sector face C_I of C , we can also define a simplex at infinity $C_I(\infty)$. Then it is known [Br1, pp. 175-177] that these simplexes at infinity give a partition of $\Delta(\infty)$ and define a spherical Tits building, denoted by Δ_∞ .

Remark 4.3.4. Another slightly different construction of the spherical Tits building Δ_∞ was contained in the proof of the Mostow strong rigidity and can be described as follows. Two sectors C_1, C_2 are called equivalent if the Hausdorff distance between them is finite. Similarly we can define equivalent sector faces, and the incidence relation between the equivalence classes of sectors and sector faces. Then the set of equivalence classes of sectors and sector faces form the spherical Tits building at infinity Δ_∞ .

If Δ is the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ of a linear connected simply connected algebraic group \mathbf{G} defined over a local field $k_{\mathfrak{p}}$, then the spherical building Δ_∞ is exactly the spherical Tits building $\Delta(\mathbf{G})$. In this case, the subset topology on $\Delta(\infty)$ induces a topology on $\Delta(\mathbf{G})$ and is different from the canonical topology of $\Delta(\mathbf{G})$ defined by the Tits metric. For example, when the rank of \mathbf{G} over $k_{\mathfrak{p}}$ is equal to 1, the Tits metric defines the discrete topology on $\Delta(\mathbf{G})$. On the other hand, the topology on $\Delta(\infty)$ is not discrete; in fact, it is a compact space homeomorphic to the Cantor set.

The topology of $\Delta(\mathbf{G})$ induced from the subset topology of $\Delta^{BT}(\mathbf{G})(\infty)$ can be described as follows. Since $\mathbf{G}(k_{\mathfrak{p}})$ acts isometrically on $\Delta^{BT}(\mathbf{G})$, it preserves the equivalence classes of rays and acts continuously on the compactification $\Delta^{BT}(\mathbf{G}) \cup \Delta(\mathbf{G})$, and hence on $\Delta(\mathbf{G})$ as well. Let σ be a simplex in $\Delta(\mathbf{G})$ of the top dimension, i.e., of dimension $r - 1$. Then the map

$$\mathbf{G}(k_{\mathfrak{p}}) \times \sigma \rightarrow \Delta(\mathbf{G})$$

is surjective. The subset topology of σ is clearly equal to the one given by the simplicial structure. In [BoS1], the topology of $\Delta(\mathbf{G})$ is given by the quotient

topology of $\mathbf{G}(k_{\mathfrak{p}}) \times \sigma$ under the above map and is shown to be compact and Hausdorff. The above discussions show that this topology is as fine as the topology induced from $\Delta^{BT}(\infty)$. Since both topologies are compact and Hausdorff, they agree.

Remark 4.3.5. The topology on $\Delta(\mathbf{G})$ naturally makes it a topological building according to Definition 3.3.1. In fact, the set Δ_1 of the vertices is equal to the disjoint union of $\mathbf{G}(k_{\mathfrak{p}})/\mathbf{P}_m(k_{\mathfrak{p}})$, $m = 1, \dots, r$, each is given with the quotient topology and is a compact Hausdorff space, where $\mathbf{P}_1, \dots, \mathbf{P}_r$ are representatives of conjugacy classes of maximal parabolic subgroups. For any $s = 2, \dots, r$, Δ_s is a finite union of $\mathbf{G}(k_{\mathfrak{p}})/\mathbf{P}(k_{\mathfrak{p}})$, where \mathbf{P} run over some representatives of conjugacy classes of parabolic subgroups of \mathbf{G} of rank s . With the quotient topology, $\mathbf{G}(k_{\mathfrak{p}})/\mathbf{P}(k_{\mathfrak{p}})$ is clearly compact and Hausdorff. Since the map $\Delta_s \rightarrow (\Delta_1)^s$ is continuous and injective, and all the spaces are compact and Hausdorff, it is clear that the image is closed. This gives the structure of a topological building on $\Delta(\mathbf{G})$.

4.4 Satake compactifications of Bruhat-Tits buildings

One of the basic points of this paper is that Euclidean buildings are analogues of symmetric spaces of noncompact type. For a symmetric space $X = G/K$ of noncompact type, we have discussed two kinds of compactifications: the geodesic compactification and the Satake compactifications. There are many other compactifications of symmetric spaces. See [BJ1] for details.

For the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ of a reductive algebraic group over a local field F , the corresponding Satake compactifications have been constructed in [ReTW2] [ReTW1] [Wer2] [Wer3] [Wer4].

The paper [GuR] compactifies the set of vertices of the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ using the space of compact subgroups of $\mathbf{G}(F)$. This interpretation of boundary points in terms of closed subgroups is satisfying because it provides a virtual geometric parametrization of maximal amenable subgroups. On the other hand, this does not give a compactification of the whole building $\Delta^{BT}(\mathbf{G})$.

The paper [ReTW1] constructs a family of finitely many compactifications of $\Delta^{BT}(\mathbf{G})$ by using Berkovich analytic geometry over complete non-Archimedean fields, and the paper [ReTW2] uses irreducible representations of the algebraic group \mathbf{G} to construct compactifications of $\Delta^{BT}(\mathbf{G})$. These compactifications are similar to the family of finitely many Satake compactifications of symmetric spaces of noncompact type. For example, the boundary can be decomposed into pieces which are Bruhat-Tits buildings of smaller algebraic groups.

The construction in [Wer2] is also similar to the Satake compactifications of symmetric spaces. Compactifications of some special buildings were treated in [Wer3] and [Wer4].

The boundary $\partial \overline{X}_{\max}^S$ of the maximal Satake compactification is naturally decomposed into symmetric spaces of noncompact type of smaller dimension, which are naturally parametrized by proper parabolic subgroups of G , where $X = G/K$, or by simplices of the Tits building $\Delta(X)$. Since each boundary symmetric space

is a cell, this shows that the boundary $\partial\overline{X}_{\max}^S$ has a natural cell structure which is dual to the Tits building. See [GJT]. The boundary $\partial\overline{X}_{\max}^S$ can also be constructed in this way by adding these boundary components using the idea of the Borel-Serre compactification of locally symmetric spaces. See [BJ1]. For Bruhat-Tits buildings, a similar construction should also work. For the Bruhat-Tits building of $\mathrm{PGL}(n)$ over a local compact field, such a construction has been carried out in [Wer2].

5 Applications of Euclidean buildings

In this section, we discuss applications of Euclidean buildings to super-rigidity of discrete subgroups of co-finite volume in the isometry groups of the quaternionic hyperbolic spaces and the Cayley hyperbolic plane, and integral Novikov conjectures for S -arithmetic groups.

The first concerns the proof by Garland of a conjecture of Serre on vanishing of certain cohomology groups of uniform discrete subgroups of p -adic Lie groups.

The second is the proof by Gromov and Schoen of the non-archimedean super-rigidity of cofinite discrete subgroups of $Sp(1, n)$ and $F_{4(-20)}$. To motivate this, we recall the Margulis super-rigidity in the higher rank case. Since the latter part of the arguments in [GS] follows the proof in [Co], we briefly outline the proof in [Co] for archimedean super-rigidity of the cofinite discrete subgroups of $Sp(1, n)$ and $F_{4(-20)}$, and comment on their differences which lead to the boundedness of the image of the representation.

We conclude this section by an application to the integral Novikov conjectures for S -arithmetic subgroups.

5.1 p -adic curvature and vanishing of cohomology of lattices

In [Kaz], Kazhdan introduced the important notion of Property (T) for a locally compact and second countable group G . Briefly, a group G has the Property (T) if every unitary representation of G which almost has invariant vectors does have nontrivial invariant vectors, i.e., the trivial representation is isolated in the unitary dual of G .

The Property (T) has been very useful for many different problems, in particular for problems around rigidity of discrete subgroups of Lie groups. See [Zi, Chap. 7] for the precise definition and applications.

In [Wa], Wang showed that if G has property (T) and π is a unitary representation of G , then $H^1(G, \pi) = 0$.

In [Kaz], Kazhdan proved that connected semisimple Lie groups with finite center and no simple factor of rank 1 have Property (T). Furthermore, any lattice subgroups Γ , i.e., discrete subgroups of finite covolume, of such Lie groups have Property (T) as well. In particular, for such lattice subgroups, $H^1(\Gamma, \pi) = 0$ for any unitary representation π of Γ . This vanishing of $H^1(\Gamma, \pi)$ is closely related to the (infinitesimal) rigidity of Γ , or rather the associated locally symmetric space $\Gamma\backslash X$, where $X = G/K$, K being a maximal compact subgroup of G (see [Mos,

p.5] for a history of results leading to Theorem 3.4.1 in §3.4, and results in [We1] [We2] [We3] [CV] [Mat1] [Mat2] [Ra1] [Ra2] [JoM] which are related to the first cohomology of Γ).

Let k be either a number field or a function field of an algebraic curve over a finite field, and $k_{\mathfrak{p}}$ the completion at a finite place. Then $k_{\mathfrak{p}}$ is a local field.

Let \mathbf{G} be a simple linear algebraic group defined over $k_{\mathfrak{p}}$, and $\Gamma \subset \mathbf{G}(k_{\mathfrak{p}})$ a torsion free discrete and cocompact subgroup.

Remark 5.1.1. It should be pointed out that when the characteristic of $k_{\mathfrak{p}}$ is equal to zero, then any co-finite discrete subgroup Γ of $\mathbf{G}(k_{\mathfrak{p}})$ is uniform. In fact, if not, it will contain a nontrivial unipotent element, which will generate a non-discrete subgroup. The assumption on the characteristic of $k_{\mathfrak{p}}$ is necessary.

Assume that the $k_{\mathfrak{p}}$ -rank r of \mathbf{G} is at least 2, then it is also known that Γ has Property (T) (see [Mar, Chap. III, Theorem 5.3] [HV]); in particular,

$$H^1(\Gamma, \pi) = 0 \tag{5.1.1}$$

for any unitary representation π of G .

Motivated by this and his computations about signs of the Euler characteristic of Γ in [Se2], Serre conjectured that for any i , $0 < i < r$,

$$H^i(\Gamma, \mathbb{R}) = 0.$$

When the rank $r = 2$, this is reduced to Equation 5.1.1.

In [Gar], Garland proved this conjecture by using the Bruhat-Tits building Δ^{BT} . (In [Gar], the cardinality of the residue field was assumed to be sufficiently large. This additional assumption was removed by Casselman in [Cas]). We will very briefly outline the idea of the proof.

Recall that for any group Γ (with discrete topology), its $K(\Gamma, 1)$ -space is defined to be a connected CW-space such that

$$\pi_1(K(\Gamma, 1)) = \Gamma, \quad \pi_i(K(\Gamma, 1)) = \{1\}, \quad i \geq 2.$$

Then it is known that

$$H^i(\Gamma, \mathbb{R}) = H^i(K(\Gamma, 1), \mathbb{R}). \tag{5.1.2}$$

Since it is often difficult to compute $H^i(\Gamma, \mathbb{R})$ directly from the definition, it is crucial to find a good model of $K(\Gamma, 1)$.

If Γ acts freely on some contractible space Y , then $\Gamma \backslash Y$ is a $K(\Gamma, 1)$ -space.

Assume that Γ is a torsion free discrete subgroup of a connected semisimple Lie group G . Let K be a maximal compact subgroup of G , and $X = G/K$ the associated symmetric space of noncompact type, which is diffeomorphic to a Euclidean space and hence contractible. Then G acts on X by the left multiplication, and the stabilizer in G of every point in X is a compact subgroup of G . This implies that $\Gamma \backslash X$ is a $K(\Gamma, 1)$ -space.

In the case under consideration, $\mathbf{G}(k_{\mathfrak{p}})$ is a non-archimedean Lie group, and there is more than one conjugacy class of maximal compact subgroups K ; and for each of them, the quotient $\mathbf{G}(k_{\mathfrak{p}})/K$ is certainly not contractible. On the other

hand, the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is a contractible simplicial complex, and $\mathbf{G}(k_{\mathfrak{p}})$ acts simplicially on it such that the stabilizer in $\mathbf{G}(k_{\mathfrak{p}})$ of every point is a compact subgroup. By assumption, Γ is a discrete, cocompact subgroup of $\mathbf{G}(k_{\mathfrak{p}})$. This implies that Γ acts freely on $\Delta^{BT}(\mathbf{G})$ if Γ is torsion free and discrete, and hence $\Gamma \backslash \Delta^{BT}(\mathbf{G})$ is a $K(\Gamma, 1)$ -space. Furthermore, since the quotient of $\Delta^{BT}(\mathbf{G})$ by $\mathbf{G}(k_{\mathfrak{p}})$ consists of a closed chamber, it follows that $\Gamma \backslash \Delta^{BT}(\mathbf{G})$ is a finite CW-complex, which gives a so-called finite $K(\Gamma, 1)$ -space.

So the conjecture of Serre is reduced to proving vanishing of $H^i(\Gamma \backslash \Delta^{BT}(\mathbf{G}), \mathbb{R})$, $0 < i < r$. In [Gar1], Garland introduced a scalar product on the finite dimensional cochain complex, and the associated Laplace operator and hence the harmonic cocycles. The analogue of the Hodge decomposition shows that $H^i(\Gamma \backslash \Delta^{BT}(\mathbf{G}), \mathbb{R})$ is isomorphic to the space of harmonic cocycles. The vanishing of these harmonic cocycles depends on lower bounds of eigenvalues of certain quadratic forms. This approach was suggested by the modification of the Bochner identity by Matsushima in [Mat1]. Since a corresponding quadratic form in [Mat1] depends on the curvature operator of the symmetric space, Garland called his operator p -adic curvature.

5.2 Super-rigidity and harmonic maps into Euclidean buildings

There have been several important generalizations of the Mostow strong rigidity. One is the Margulis super-rigidity [Mar], and another is for more general, not necessarily locally symmetric spaces by the approach of using harmonic maps and Bochner type formulas, which was initially suggested by S.T.Yau [Yau1, p. 37] to study both rigidity of discrete subgroups and complex structures of certain manifolds.

In fact, the Mostow rigidity was used by Yau to prove his rigidity of complex structures on $\mathbb{C}P^2$ as a corollary of his celebrated solution of Calabi's conjecture [Yau4, Theorem 6].

Theorem 5.2.1. *Let N be a compact quotient of the unit ball in \mathbb{C}^2 by a (uniform) lattice in $SU(1, 2)$. Then any complex surface homotopy equivalent to N preserving the canonical orientations induced from the complex structures is biholomorphic to N .*

In the proof of this theorem, the Mostow strong rigidity was used. This is the first instance where the Mostow rigidity for locally symmetric spaces is combined with the rigidity of complex structures of Kähler manifolds (see also [Yau4, Remark after Theorem 5] on rigidity of the complex structure of $\mathbb{C}P^n$).

Motivated by these results, Yau suggested the idea of using harmonic maps to study both the rigidity of discrete subgroups and the rigidity of complex structures of Kähler manifolds with negative curvature [Yau1, p. 37] [Yau3] [JY3]. In [Siu2] [Siu1], Siu carried out this approach for the rigidity of the complex structures with a slightly stronger condition on negativity of the curvature.

The Siu-Yau method of using harmonic maps is very elegant and has also turned out to be extremely powerful for both Mostow type rigidity of lattices

of Lie groups and the rigidity of geometric structures (see [JY1] [JY3] [MSY] [Mok1] [Mok2] and the references there). In fact, it can be used to prove the Mostow strong rigidity and the Margulis super-rigidity in many cases ([MSY] [JY1] [JZ]). More importantly, it can be used to prove new cases of super-rigidity, i.e., the super-rigidity of lattices acting on the quaternionic hyperbolic spaces and the Cayley hyperbolic plane, which can not be proved by other methods. More specifically, Corlette proved the archimedean super-rigidity of such lattices in [Co], and Gromov and Schoen developed a theory of harmonic maps to the Bruhat-Tits buildings and used it to prove the non-archimedean super-rigidity of these lattices in [GS].

Remark 5.2.2. The generalization of the Mostow strong rigidity to \mathfrak{p} -adic Lie groups and lattices is given in [Pr2]. Another generalization of rigidity for lattices in $PGL(d, k)$, where k is a local non-Archimedean field, is given in [Ber1]. A generalization to graphs, in particular to quotients of Bruhat-Tits trees, of the entropy rigidity of rank 1 locally symmetric spaces in [BCG1] is given in [Lim].

Remark 5.2.3. Some other versions of rigidity of lattices in $PU(1, 2)$, for example, the super-rigidity of lattices satisfying some cohomological conditions and integrality of lattice subgroups, are given in [Klig1] and [Ye1] [Ye2]. More precisely, let $\mathbf{G} \subset GL(n)$ be an algebraic group defined over \mathbb{Q} . A subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ is called *integral* if it contains a subgroup Γ' of finite index such that $\Gamma' \subset \mathbf{G}(\mathbb{Q}) \cap GL(n, \mathbb{Z})$. If Γ' is also of finite index in $\mathbf{G}(\mathbb{Q}) \cap GL(n, \mathbb{Z})$, then Γ is an arithmetic subgroup. The symmetric space of noncompact type $X = PU(1, 2)/U(2)$ can be identified with the complex unit ball $B_{\mathbb{C}}^2$ in \mathbb{C}^2 . One result in [Ye1] says that if a ball quotient $\Gamma \backslash B_{\mathbb{C}}^2$ has the first Betti number equal to 1 and the Picard number equal to 1, then Γ is integral. Under other conditions, the results in [Ye2] and [Klig1] show that Γ is arithmetic. In the proofs of these results, Bruhat-Tits buildings played an important role. See also [PYe1] [PYe3] [CaS1] for applications of Bruhat-Tits buildings in classifying fake projective planes, where a fake projective plane is a smooth compact complex surface which is not the complex projective plane but has the same first and second Betti numbers as the complex projective plane. See also [PYe2] for related results on fake P^4 spaces.

Remark 5.2.4. The idea of using harmonic maps is very fruitful. Besides the applications in [Co] and [GS], it has also been used by Sampson [Sa], Simpson [Sim], Carlson-Toledo [CT] and others. As well-known, the Mostow strong rigidity fails for hyperbolic Riemann surfaces. On the other hand, the theory of harmonic maps between Riemann surfaces of negative curvature developed by Schoen and Yau [SY] [Yau3] has been very useful in studying the Teichmüller spaces and related problems [Wol1] [Wol2] [Ru] [DDW].

In this subsection, we briefly discuss the results in [Co] and [GS]. First, the Mostow strong rigidity in Theorem 3.4.1 can be restated as follows.

Proposition 5.2.5. *Let G, G' be two connected semisimple Lie groups with trivial center and no compact simple factors. Suppose that $\Gamma \subset G, \Gamma' \subset G'$ are uniform lattices, Γ is irreducible. Suppose furthermore that if G is of rank one, then G is not*

equal to $SL(2, \mathbb{R})$, or equivalently $\dim G \geq 4$. Then any isomorphism $\pi : \Gamma \rightarrow \Gamma'$ extends to an isomorphism $\pi : G \rightarrow G'$.

The super-rigidity removes the assumption that the image $\pi(\Gamma)$ is a uniform lattice in G' [Zi, Theorem 5.1.2] [Mar, Theorems 2 and 3, pp. 2-3].

Theorem 5.2.6. *Let \mathbf{G} be a connected linear semisimple algebraic group defined over \mathbb{R} with \mathbb{R} -rank at least 2. Assume that the identity component $\mathbf{G}(\mathbb{R})^0$ has no compact factors. Suppose that $k_{\mathfrak{p}}$ is a local field of characteristic 0, and \mathbf{H} a connected linear algebraic group defined and simple over $k_{\mathfrak{p}}$. Assume that $\pi : \Gamma \rightarrow \mathbf{H}(k_{\mathfrak{p}})$ is a homomorphism with $\pi(\Gamma)$ Zariski dense in \mathbf{H} . Then one of the following alternatives hold:*

1. *If $k_{\mathfrak{p}} = \mathbb{R}$ and $\mathbf{H}(\mathbb{R})$ is not compact, then π extends to a rational homomorphism $\mathbf{G} \rightarrow \mathbf{H}$ defined over \mathbb{R} .*
2. *If $k_{\mathfrak{p}} = \mathbb{C}$, then either (a) the closure of $\pi(\Gamma)$ in the ordinary topology is compact, or (b) π extends to a rational homomorphism $\pi : \mathbf{G} \rightarrow \mathbf{H}$.*
3. *If $k_{\mathfrak{p}}$ is totally disconnected (i.e., non-archimedean), then $\pi(\Gamma)$ is bounded in $\mathbf{H}(k_{\mathfrak{p}})$, i.e., its closure with respect to the natural Hausdorff topology of $\mathbf{H}(k_{\mathfrak{p}})$ is compact.*

An important consequence of this theorem is the celebrated arithmeticity theorem of irreducible lattices of rank 2 (see [Zi, Theorem 6.1.2] [Mar, Theorem 1, p. 2, Theorem (1'), p. 4]), which together with a result of Prasad [Pr3] show that Proposition 5.2.5 holds when Γ, Γ' are only assumed to be lattices, but not necessarily co-compact. It should be pointed out that if $\mathbf{G}(\mathbb{R})$ is the real locus of a linear connected semisimple algebraic group defined over \mathbb{R} and \mathbf{G} has no compact simple factors, then the Borel density [Zi, Theorem 3.2.5] says that any lattice Γ in $\mathbf{G}(\mathbb{R})$ is Zariski dense in \mathbf{G} , i.e., Γ is dense in \mathbf{G} with respect to the Zariski topology. So the assumption on the Zariski density in Theorem 5.2.6 is natural.

Since the strong Mostow rigidity in Proposition 5.2.5 holds in the rank one case with one exception, it is natural to ask if the super-rigidity holds for lattices in some rank one Lie groups as well.

According to the classification of symmetric spaces of (\mathbb{R}) -rank 1 (see [Mos, §19]), there are four types:

1. the real hyperbolic spaces $\mathbf{H}_{\mathbb{R}}^n = SO^0(1, n)/SO(n)$, $n \geq 2$;
2. the complex hyperbolic spaces $\mathbf{H}_{\mathbb{C}}^n = SU(1, n)/U(n)$, $n \geq 2$;
3. the quaternionic hyperbolic spaces $\mathbf{H}_{\mathbb{H}}^n = Sp(1, n)/Sp(n)$, $n \geq 2$;
4. the (exceptional) Cayley hyperbolic plane $\mathbf{H}_{\mathbb{O}}^2 = F_{4(-20)}/Spin(9)$.

It is known [Co, p.166] that the super-rigidity for lattices Γ acting on the real and complex hyperbolic spaces, $\Gamma \subset SO^0(1, n)$ or $SU(1, n)$, does not hold due to existence of non-arithmetic lattices. In [Co], it is shown that the archimedean super-rigidity, i.e., $k_{\mathfrak{p}} = \mathbb{R}, \mathbb{C}$, holds for lattices in $Sp(1, n)$ and $F_{4(-20)}$. It uses

the method of harmonic maps initiated in [Si2] (see [JY1] for a survey of the Bochner-Matsushima type formulas and [Mat1] [Ye3]).

The proof of the Mostow strong rigidity theorem using harmonic maps goes in two steps:

1. Let ψ be the map in the Equation 3.4.1. Using the nonpositive curvature of the target manifold, show that there exists a unique harmonic map $\Psi : \Gamma \backslash X \rightarrow \Gamma \backslash X'$ in the homotopy equivalence class of the map ψ .
2. Using a Bochner type formula, show the vanishing of certain derivatives of Ψ and hence that Ψ is totally geodesic, which in turn implies that Ψ is an isometry up to suitable scaling.

Since the harmonic map Ψ is deformed from ψ (or rather the equivariant quasi-isometry φ), this gives explicitly an isomorphism between the Lie groups G and G' in Theorem 3.4.1. The arguments in the other cases are similar, replacing the harmonic maps between $\Gamma \backslash X$ and $\Gamma \backslash X'$ by equivariant maps. We briefly outline the arguments in [Co] and indicate where the assumption that the Lie groups are $Sp(1, n)$ or $F_{4(-20)}$ is used.

Let $G = Sp(1, n)$ or $F_{4(-20)}$, and $\Gamma \subset G$ a lattice. Let G' be a simple noncompact Lie group with trivial center, and $\rho : \Gamma \rightarrow G'$ a homomorphism with a Zariski dense image. We need to show that ρ extends to a homomorphism $\rho : G \rightarrow G'$. There are three steps:

1. There is a finite-energy, ρ -equivariant map from $X = G/K$ to $X' = G'/K'$, where $K \subset G$ and $K' \subset G'$ are maximal compact subgroups.
2. Show that the finite-energy equivariant map can be modified into a finite-energy ρ -equivariant harmonic map $f : X \rightarrow X'$.
3. On X , there exists a parallel differential form ω such that its annihilator in $\text{End}(T_o X)$ consists of only skew-symmetric endomorphisms, where $T_o X$ is the tangent space to X at $o \in X$.
4. The harmonic map f is totally geodesic and the Zariski density of the image of Γ implies that f is an isometry between X and X' , which in turn implies that ρ extends to a homomorphism from G to G' .

Step (1) follows from [Co, Proposition 2.3] and depends on the fact that the root spaces have sufficiently large dimensions, which is satisfied by $Sp(1, n)$ and $F_{4(-20)}$. Step (2) is given in [Co, Theorem 2.1] and works for other groups as well. Step (3) is proved in [Co, Propositions 1.2 and 1.4], and also works for some other higher rank groups [Co, p. 166]. Step (4) is given in [Co, Proposition 3.3] and depends crucially on Step (3) and a new Bochner-type formula, i.e., $D^*(df \wedge \omega) = 0$ in [Co, Theorem 3.2].

The arguments in [GS] are similar to the arguments in [Co] once the theory of harmonic maps into Euclidean buildings (or rather F -connected complexes) is established. We indicate briefly why in this case the image of the representation

of Γ is bounded, while in [Co], the representation of Γ extends to an isomorphism of the Lie group and hence the image is clearly not bounded.

In fact, let $G = Sp(1, n)$ or $F_{4(-20)}$ and $\Gamma \subset G$ a lattice as above, and \mathbf{G}' be an almost simple algebraic group defined over a non-archimedean local field $k_{\mathfrak{p}}$. Let $\rho : \Gamma \rightarrow \mathbf{G}'(k_{\mathfrak{p}})$ be a homomorphism. Then we need to show the image $\rho(\Gamma)$ is a bounded subgroup in $\mathbf{G}'(k_{\mathfrak{p}})$. Let $\Delta^{BT}(\mathbf{G}')$ be the Bruhat-Tits building associated with \mathbf{G}' (or rather to its simply connected covering group) in §4.2.

The proof in [GS] follows several steps:

1. The homomorphism ρ induces a finite-energy Lipschitz ρ -equivariant map from $X = G/K$ to $\Delta^{BT}(\mathbf{G}')$.
2. There is a finite-energy Lipschitz ρ -equivariant map of least energy $u : X \rightarrow \Delta^{BT}(\mathbf{G}')$ whose restriction to small balls at any point is also minimizing, which implies that u is a harmonic map.
3. The map u is called differentiable (or non-singular) at a point $p \in X$ if the local images of u around p are contained in an apartment of $\Delta^{BT}(\mathbf{G}')$. Show that the set of singular points of u has Hausdorff dimension at most $\dim X - 2$.
4. Show that the harmonic map u is constant around the differentiable points and is hence constant.
5. Let p be the unique point in the image $u(X)$. Then the image $\rho(\Gamma)$ is contained in the stabilizer in $\mathbf{G}'(k_{\mathfrak{p}})$ of the point $p \in \Delta^{BT}(\mathbf{G}')$, and is hence contained in a parahoric subgroup of $\mathbf{G}'(k_{\mathfrak{p}})$, which is a bounded subgroup.

Step (1) is proved in [GS, Lemma 8.1] and uses $\dim X \geq 8$. Step (2) is proved in [GS, Theorem 7.1]. Step (3) is proved in [GS, Theorem 6.4], which works for more general F -connected complexes.

Step (4) is given in [GS, Theorem 7.4] and is a new feature not present in [Co]. In fact, since the local image is contained in an apartment which can be identified with \mathbb{R}^r , where r is the $k_{\mathfrak{p}}$ -rank of \mathbf{G}' , the coordinates of the harmonic map u give parallel 1-forms. If they are not zero, they will give local isometric splitting of X with \mathbb{R} as a factor. Since this is impossible, it implies that u must be locally constant.

Remark 5.2.7. The philosophical differences between the original approach of Mostow and the approach here are interesting. In the former case, the quasi-isometry φ , the lift of ψ , is pushed to infinity in X and X' to get precise maps, i.e., an isomorphism of the spherical Tits buildings, and a homeomorphism of the maximal Furstenberg boundaries. On the other hand, in the latter case, the quasi-isometry φ is deformed locally into an isometry via the theory of harmonic maps. In a sense, the former is a global argument, and the latter is local and does not depend on the theory of spherical Tits buildings. It is also instructive to point out that the idea in §4 is to scale down or shrink symmetric spaces infinitely to get \mathbb{R} -Euclidean buildings. This is another type of global geometry. These remarks confirm the point of view that buildings reflect the global geometry in more than one way.

Remark 5.2.8. After the success of [GS] in applying the theory of harmonic maps into Euclidean buildings, the theory has been generalized to CAT(0)-metric spaces. See [KS1] [KS2] [Jo3] [Bal1] [JZ].

Remark 5.2.9. The Mostow strong rigidity deals with uniform lattices in semisimple Lie groups. Its generalization to lattices in semisimple p -adic Lie groups is given by Prasad in [Pr2]. The proof followed the method of [Mos1]. Instead of realizing the spherical Tits building using the geometry at infinity of the symmetric spaces of noncompact type, Euclidean buildings were used to obtain spherical buildings at infinity, which will be explained in the next subsection.

Remark 5.2.10. In [Mar], lattice subgroups in products of real and p -adic groups, in particular, S -arithmetic subgroups, are studied, and hence the archimedean and non-archimedean places are studied at the same time. Basically, irreducible S -arithmetic subgroups of semisimple algebraic groups of total rank greater than or equal to 2 are super-rigid.

Remark 5.2.11. There is also a version of Mostow strong rigidity for non-Archimedean uniformization in [IsK].

5.3 Applications to S -arithmetic groups

In this subsection, we briefly recall two applications of the Bruhat-Tits buildings to S -arithmetic groups. The first concerns the cohomology groups, and the second one the integral Novikov conjectures.

First we define S -arithmetic subgroups. Let k be a global field, i.e., either a number field, a finite extension of \mathbb{Q} , or the function field of a smooth projective curve over a finite field, for example $\mathbb{F}_p(t)$, where p is a prime.

Let S be a finite set of places of k including the set S_∞ of all infinite places. Let $S_f = S - S_\infty$ be the subset of finite places. For each finite place \mathfrak{p} , let $\nu_{\mathfrak{p}}$ be the corresponding valuation. Define the ring \mathcal{O}_S of S -integers in k by

$$\mathcal{O}_S = \{x \in k \mid \nu_{\mathfrak{p}}(x) \geq 0, \mathfrak{p} \notin S\}.$$

Let $\mathbf{G} \subset GL(n)$ be a linear algebraic group defined over k . A subgroup Γ of $\mathbf{G}(k)$ is called an S -arithmetic subgroup if it is commensurable with $\mathbf{G}(\mathcal{O}_S) = \mathbf{G}(k) \cap GL(n, k)$.

Assume that \mathbf{G} is semisimple. Then Γ is a co-finite discrete subgroup in

$$G_S = \prod_{\mathfrak{p} \in S} \mathbf{G}(k_{\mathfrak{p}}),$$

where for $\mathfrak{p} \in S_f$, $k_{\mathfrak{p}}$ is the completion of k with respect to the norm associated with the valuation $\nu_{\mathfrak{p}}$, and for $\mathfrak{p} \in S_\infty$, $k_{\mathfrak{p}}$ is the completion of k with respect to the associated archimedean norm. It is also known that Γ is uniform if and only if the k -rank r of \mathbf{G} is equal to 0.

Proposition 5.3.1. (1) When k is a number field, any S -arithmetic subgroup Γ admits a torsion free subgroup of finite index. (2) When k is a function field, if

the rank $r = 0$, any S -arithmetic subgroup admits a torsion free subgroup of finite index; on the other hand, if the rank $r > 0$, Γ does not admit any torsion free subgroup of finite index.

(1) is proved in [Se2, p. 126, (a₃)] (see also [Br1, p. 196]). The first part of (2) is proved in [Se2, p. 126, Case (b)]. The second part follows from the fact that any subgroup of finite index of Γ contains unipotent elements which are torsion since the characteristic of k is positive.

An important ingredient in studying cohomology groups and related cohomological finiteness of discrete subgroups Γ is finding or constructing good models of $K(\Gamma, 1)$ -spaces (see [Br3])

Remark 5.3.2. In Proposition 5.3.1, when the rank $r = 0$, S -arithmetic subgroups of algebraic groups over function fields are torsion-free and their infinite index torsion-free subgroups admit finite classifying spaces. In the rank $r > 0$, they do not admit torsion-free finite index subgroups and hence do not admit any finite classifying spaces, or even finite dimensional classifying spaces.

Recall that a group Γ is said to be of type F_m if it admits a classifying space with finite m -skeleton. The *finiteness length* of Γ is the largest integer m such that Γ is of type F_m .

For S -arithmetic subgroups of algebraic groups over function fields of positive rank, *an important problem is to determine the finiteness length*. A lot of important work has been done recently. See [BuGW1] [BuGW2] [BuW] [Bux] [Abe1] [Abe2] [Abr1] [Abr2] for precise statements of results and a history of and references on this problem.

From now on, we assume that \mathbf{G} is semisimple. When $S = S_\infty$, Γ is an arithmetic subgroup, and G_S is a Lie group. Let $X_\infty = G_S/K$ be the associated symmetric space of noncompact type. Then Γ acts properly on X_∞ . If Γ is torsion free, then $\Gamma \backslash X_\infty$ is a $K(\Gamma, 1)$ -space.

Otherwise, for every $\mathfrak{p} \in S_f$, let $X_{\mathfrak{p}}$ be the Bruhat-Tits building of \mathbf{G} when considered as an algebraic group defined over $k_{\mathfrak{p}}$. Denote the symmetric space associated with S_∞ by X_∞ as before. Set

$$X_S = X_\infty \times \prod_{\mathfrak{p} \in S_f} X_{\mathfrak{p}}.$$

Then Γ acts properly on X_S , and hence the stabilizer in Γ of any point in X_S is finite. If Γ is torsion free, then Γ acts freely on X_S . Since X_∞ and each $X_{\mathfrak{p}}$ is contractible, this implies that $\Gamma \backslash X_S$ is a $K(\Gamma, 1)$ -space in this case.

When the rank $r > 0$, $\Gamma \backslash X_S$ is not compact and hence is not a finite CW -complex, i.e., not a finite $K(\Gamma, 1)$ -space. If k is a number field, we can enlarge X_S to

$$\overline{X}_S^{BS} = \overline{X}_\infty^{BS} \times \prod_{\mathfrak{p} \in S_f} X_{\mathfrak{p}},$$

where \overline{X}_∞^{BS} is the Borel-Serre partial compactification in §2.12. (We can use the functor of restriction of scalars to replace \mathbf{G} by an algebraic group defined over \mathbb{Q} so

that the construction in §2.12 applies). Then $\Gamma \backslash \overline{X}_S^{BS}$ is compact and can be shown to have the structure of finite CW-complex. Since the inclusion $\Gamma \backslash X_S \rightarrow \Gamma \backslash \overline{X}_S^{BS}$ is a homotopy equivalence, this gives a concrete finite $K(\Gamma, 1)$ -space.

The Γ -action on X_S and \overline{X}_S^{BS} and their quotients have played an important and fundamental role in understanding cohomology groups and other related finiteness properties of Γ . The compactification $\Delta^{BT}(\mathbf{G}) \cup \Delta(\mathbf{G})$ discussed in the previous subsection is also crucial in such applications, which are discussed in great detail in surveys [Se2] [Br1, Chap. VII] [Ro2, §10] [RS] (see also the original paper [BoS1]).

Remark 5.3.3. Cohomological finiteness properties of S -arithmetic subgroups of algebraic groups defined over a global field depend in an essential way on whether the field is a number field or a function field, i.e., the function field of a smooth curve over a finite field. In the latter case, the many different kinds of cohomological finiteness properties of S -arithmetic subgroups are not well understood. See [Abe1] [Abe2] [AbA] [Beh1] [Beh2] [Bux] and [Br1, Chap. VII, §4]. In all these works, actions on Bruhat-Tits buildings or suitable subcomplexes play an important role.

Remark 5.3.4. As we pointed out in Remark 4.3.5, the boundary $\Delta(\mathbf{G})$ in the compactification $\Delta^{BT}(\mathbf{G}) \cup \Delta(\mathbf{G})$ is a topological spherical Tits building rather than a usual spherical Tits building. Besides the application to the problem of rank rigidity and the rigidity of isoparametric submanifolds, this also confirms our point of view that in applications to geometry and topology, the topological buildings are more natural.

In the rest of this subsection, we discuss an application to the integral Novikov conjectures for S -arithmetic subgroups Γ .

To explain this, we start with the original Novikov conjecture on the homotopy invariance of higher signatures (see [FRR] for precise statements and history of the Novikov conjectures). The Hirzebruch index theorem says that the signature of a compact oriented manifold M of dimension $4n$ is equal to the evaluation of the Hirzebruch class \mathcal{L}_M on the fundamental class, which implies that the latter is an oriented homotopy invariant. Novikov introduced higher signatures by the evaluation of \mathcal{L}_M cupped with some cohomology classes pulled back from a $K(\pi, 1)$ -space, where $\pi = \pi_1(M)$, which are called higher signatures, and conjectured that they are homotopy invariant.

This original Novikov conjecture is equivalent to the rational injectivity of the assembly map in the surgery (or L -) theory, and the (integral) injectivity of the assembly map in the L -theory is called the integral Novikov conjecture, which is an important step towards computing the L -groups, the image groups of the assembly map. Similarly, there are also assembly maps for the (topological) K -groups of C^* -algebras, in particular the reduced group C^* -algebras which are the ℓ^2 -completions of group algebras $\mathbb{C}[\Gamma]$ of groups Γ , and for the algebraic K -groups of the group ring $\mathbb{Z}[\Gamma]$ of groups Γ . In each theory, the rational injectivity of the assembly map is called the Novikov conjecture, and the injectivity of the assembly map is called the *integral* Novikov conjecture in that theory.

A particularly interesting class of groups Γ with discrete topology is the class of discrete subgroups of Lie groups. For torsion free arithmetic subgroups Γ of linear algebraic groups, the integral Novikov conjectures were proved in [J2] by showing that arithmetic subgroups have finite asymptotic dimension (see [J2] for a brief history about the integral Novikov conjectures for such discrete subgroups). It should be emphasized that the torsion free assumption is in general necessary for the integral Novikov conjectures.

The next natural and important class of groups is the class of S -arithmetic subgroups of linear algebraic groups defined over global fields. They are in general not contained in real Lie groups as discrete subgroups.

To prove the integral Novikov conjecture for them, we use the following general result of Carlsson and Pedersen [CP].

Proposition 5.3.5. *Suppose that Γ has a finite $K(\Gamma, 1)$ -space, denoted by $B\Gamma$, and the universal cover $E\Gamma$ of $B\Gamma$ has a contractible, metrizable Γ -compactification $\overline{E\Gamma}$ such that the action of Γ on $\overline{E\Gamma}$ is small at infinity. Then the integral Novikov conjecture holds for Γ .*

In the above proposition, by a small action at infinity we mean that for any compact subset $K \subset E\Gamma$, if $g_j K$ has an accumulation near $z \in \partial E\overline{\Gamma}$, then the whole set $g_j K$ is contained in small neighborhoods of z .

A typical example is when Γ is the fundamental group of a *closed* nonpositively curved manifold M . Then we can take $B\Gamma = M$, and $E\Gamma$ the universal covering \tilde{M} . Since \tilde{M} is a Hadamard manifold, it admits the geodesic compactification $\tilde{M} \cup \tilde{M}(\infty)$, and the Γ -action on \tilde{M} is small.

As discussed before, for a torsion free S -arithmetic subgroup Γ of a connected linear semisimple algebraic group defined over a global field k , we can take $\Gamma \backslash X_S$ to be a not necessarily finite $B\Gamma$. When the rank r is equal to zero, it can be shown to be a finite $K(\Gamma, 1)$ -space. Now $E\Gamma = X_S$. Since X_∞ is nonpositively curved and hence a complete CAT(0)-space, and each Bruhat-Tits building $X_{\mathfrak{p}}$ is also a complete CAT(0)-space, this follows that X_S is a complete CAT(0)-space. By Proposition 3.5.2 and Proposition 4.3.3, X_S admits a contractible compactification $X_S \cup X_S(\infty)$. Since Γ acts on the interior X_S with a compact quotient and the compactification $X_S \cup X_S(\infty)$ is defined in terms of equivalence classes of rays, it can be shown easily that the extended action on $X_S \cup X_S(\infty)$ is small at infinity. By Proposition 5.3.5, we obtain the following result [J1].

Proposition 5.3.6. *If the rank r of a semisimple linear algebraic group \mathbf{G} over k is equal to zero, then the integral Novikov conjecture in K -theory and L -theory holds for any torsion free S -arithmetic subgroup of \mathbf{G} .*

Perhaps it should be pointed out that the existence of a finite or even finite-dimensional $B\Gamma$ implies that Γ is torsion free. By the earlier discussions in this subsection, when k is a function field, torsion free S -arithmetic subgroups exist only in the case when the rank $r = 0$, and hence the result in the proposition is optimal in this case.

On the other hand, when k is a number field, the assumption that $r = 0$ might be removed. But the general method in Proposition 5.3.5 can not be applied

directly. In fact, when $r > 0$, we can take the compactification $\Gamma \backslash \overline{X}_S^{BS}$ as a finite $K(\Gamma, 1)$ -space and \overline{X}_S^{BS} as $E\Gamma$. To apply Proposition 5.3.5, we need to construct a small compactification of the *partial* compactification \overline{X}_S^{BS} . This does not seem to be possible. We need a generalized version of Proposition 5.3.5 by Carlsson-Pedersen with weaker conditions on $E\Gamma$. See [Gol] for the statement of this result and an application to arithmetic subgroups of \mathbb{R} -rank 1 semisimple algebraic groups.

Remarks 5.3.7. After a preliminary version of this paper was written, the author has proved a generalized integral Novikov conjecture in both L- and K-theories for all S-arithmetic subgroups, which may contain torsion elements, of \mathbf{G} of rank $r = 0$ over a global field k (see [J1]), using the method in [Rose].

It is also recently proved in [J4] that all torsion-free S-arithmetic groups Γ of a semisimple linear algebraic group \mathbf{G} over a number field k , without the rank zero restriction, has finite asymptotic dimension and finite $B\Gamma$ -spaces; hence the integral Novikov conjecture in both L- and K-theories holds for them. It is not known if the generalized integral Novikov conjecture holds if the S-arithmetic groups contain torsion elements.

5.4 Applications to harmonic analysis and representation theories

For semisimple algebraic groups \mathbf{G} over local fields F , construction of the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ is only part of the Bruhat-Tits theory for \mathbf{G} , which describes in detail various subgroups and decompositions and associated integral algebraic group schemes. In fact, the geometry of the Bruhat-Tits buildings allows one to construct various filtrations of subgroups and to understand geometrically (or visually) and better relations between these subgroups subgroups. These structures on subgroups of $\mathbf{G}(F)$, their subgroups and the associated algebraic group schemes are fundamental to harmonic analysis and representation theory of $\mathbf{G}(F)$. To summarize, *the Bruhat-Tits theory is about structure and properties of semisimple (or more general) reductive algebraic groups \mathbf{G} over local fields*, their subgroups and related integral algebraic schemes, while *the Bruhat-Tits building* is only the underlying geometric or topological part of the Bruhat-Tits theory.

One important difference between the Bruhat-Tits building $\Delta^{BT}(\Delta)$ and the spherical Tits building $\Delta(\mathbf{G})$ of a semisimple algebraic group over a field k is that $\Delta(\mathbf{G})$ gives a reformulation of structures and properties of parabolic subgroups of \mathbf{G} defined over k . But structures and properties of subgroups of $\mathbf{G}(F)$ over local fields are developed simultaneously with the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$. In this sense, the Bruhat-Tits building is *an integral part* of the Bruhat-Tits theory.

There are many applications of the Bruhat-Tits theory, in particular the Bruhat-Tits building to harmonic analysis and representation theories. See [Yu2] for a survey and introduction to the Bruhat-Tits theory, [Yu1] for an explicit introduction to the Bruhat-Tits theory, [Ra] for an explicit description of the Bruhat-Tits building of a simply connected Chevalley group and application to classification of

depth-zero super-cuspidal representations, and the book [Ma1] on spherical harmonic analysis on p -adic groups and [Ma2] for related affine Hecke algebras.

Due to lack of knowledge of the author, we will make a list of applications with references.

1. The Prasad-Moy filtration of parahoric subgroups of reductive algebraic groups over local fields and existence of refined minimal type via the filtration for irreducible admissible representation of $\mathbf{G}(F)$ [MP1], and classification of depth zero representation [MP2].
2. Constructing and understanding representation of $\mathbf{G}(F)$ via sheaves and cohomology theories of the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ [SS1].
3. Relations to p -adic symmetric spaces, and the role of the Bruhat-Tits buildings in understanding cohomology of Drinfeld's generalized upper halfplane and its quotients by cocompact discrete groups [SS2], and applications to representation theories of p -adic groups.
4. Proof of Howe's conjecture on restriction of invariant distributions on the Hecke algebra using the geodesic convexity of the displacement function on the Bruhat-Tits building, which is related to the fact that the Bruhat-Tits building is a CAT(0)-space [BaM] [Moy].
5. Construction and completeness of supercuspidal representations [Yu3] [Kim] and other properties of supercuspidal representations [DeR1] [DeR2] [HaM].
6. Parametrization of nilpotent orbits [De1] and of conjugacy classes of maximal unramified tori [De3] via the Bruhat-Tits theory. See also [De2]. The papers [De4] [De5] give more expository discussions of these results.
7. Expansion of the character of an irreducible admissible representation [KiM] and expansion of orbital integral [Rog].
8. Good orbital integrals in Lie algebra and lifting of the Fundamental Lemma [CuH].
9. Formulation and proof of a conjecture on duality for representations of p -adic and finite reductive groups [Vig].
10. Topological central extensions of semisimple groups over local fields [PrR1] [PrR2], which motivated the Moy-Prasad filtration in [MP1].
11. Computing the volume of S -arithmetic quotients of semisimple algebraic groups [Pr1].
12. Application to Hasse principle and computation of Shafarevich-Tate group [Gi1].

Some other results on Bruhat-Tits buildings motivated by the representation theories are contained in [PYu] [KM].

Another application of Bruhat-Tits buildings to representation theory is given in [GaL]. Let \mathbf{G} be a complex semisimple group, and \mathbf{G}^\vee be the Langlands dual of \mathbf{G} . Then the character of finite dimensional highest weight irreducible modules of \mathbf{G}^\vee can be expressed in terms of galleries of the Bruhat-Tits building of the loop group $\mathbf{G}(\mathbb{C}((t)))$.

Bruhat-Tits buildings have also been used in generalizations of the classical Horn problem on eigenvalues of Hermitian matrices. Let A , B and C be three Hermitian matrices satisfying the equation $C = A + B$. If A, B commute, then it is clear that the eigenvalues of C are equal to the sum of eigenvalues of A and B . Otherwise, the eigenvalues of C are bounded by linear inequalities of the eigenvalues of A and B . The Horn problem is to find a *minimal set of explicit inequalities* for the eigenvalues of C . This problem turned out to be related to many other problems in geometric invariant theory, Schubert calculus, representation theories, and geometry of symmetric spaces and Bruhat-Tits buildings. For an introduction of this problem and related results, see [Ful].

Hermitian matrices are points of the symmetric space $GL(n, \mathbb{C})/U(n)$, and the Horn problem and related problem for weights of tensor products of finite dimensional highest weight representations can be generalized to other reductive groups, and they are also related to the geometry of triangles in symmetric spaces and Bruhat-Tits buildings. For some references on this problems after the paper of [Fu], see [HaKM] [KaLM1] [KaLM2] [BelP1] [BelP2] [Res] [Ram].

6 \mathbb{R} -trees and \mathbb{R} -buildings

The Bruhat-Tits buildings of semisimple algebraic groups defined over local fields are simplicial complexes and locally compact. In this section, we recall \mathbb{R} -trees and \mathbb{R} -Euclidean buildings. They are analogues of the Bruhat-Tits buildings for algebraic groups over fields with non-discrete valuations but not simplicial complexes or locally compact topological spaces. They also arise naturally in the study of global geometry of symmetric spaces of noncompact type and the Euclidean buildings. In fact, \mathbb{R} -trees have been used crucially in reproofs of some fundamental results of Thurston's geometrization program of three dimensional manifolds [Th1] [Th2] [Th2], and \mathbb{R} -Euclidean buildings were used to prove the Margulis conjecture on quasi-flats in symmetric spaces [KL2].

First we discuss \mathbb{R} -trees as metric spaces. Then we show how to construct them from Λ -valued metric spaces, where Λ is an ordered abelian group contained in \mathbb{R} , and $SL(2, F)$ where F is a field with non-discrete valuation. \mathbb{R} -trees have many applications in topology, and we mention only one sample result from [MS2]. \mathbb{R} -Euclidean buildings are more complicated and can not be easily described in terms of BN -pairs; and we follow the method in [BT1, §7.4] to construct them for algebraic groups. Finally we consider their applications to quasi-isometric rigidity of symmetric spaces.

6.1 Definition of \mathbb{R} -trees and basic properties

\mathbb{R} -trees were first introduced by Tits [Ti2], and a self-contained description was given in [MS1]. Surveys about \mathbb{R} -trees and their applications are given in [Mor1] [Shn]. We will recall some applications in the next subsection.

Definition 6.1.1. *A metric space (T, d) is called an \mathbb{R} -tree if every pair of points $p, q \in M$ is connected by a unique arc $c : [0, \ell] \rightarrow M$, where $\ell = d(p, q)$, such that (1) $c(0) = p$, $c(\ell) = q$; (2) the map c is an isometric embedding.*

The image of c in T is called a segment. Clearly, if two segments c_1, c_2 only intersect at one end point, their union is also a segment. The intersection of any two segments is either a point or a segment. These properties characterize \mathbb{R} -trees among geodesic metric spaces.

Roughly speaking, \mathbb{R} -trees have two characteristic properties: (1) they are 1-dimensional, (2) they don't contain any loops.

A usual (simplicial) tree with a linear distance function on every edge is clearly a \mathbb{R} -tree. To explain the difference between simplicial trees and general \mathbb{R} -trees, we note if we normalize the edge lengths of a simplicial tree to be equal to 1, then the set of vertices (or nodes or branch points) of the tree is a discrete subset. On the other hand, in a general \mathbb{R} -tree which arises in topology and geometry, the set of branch points is dense or even equal to the whole tree.

Since \mathbb{R} -trees are not 1-dimensional simplicial complexes in general, we could not completely describe them usually by specifying vertices and edges between them. On the other hand, we can start with specifying the collection of vertices and a distance on them which measures the lengths of the edges connecting them and hence determines the edges in some sense.

To make this more precise and rigorous, we briefly recall the construction of \mathbb{R} -trees [MS1]. Let Λ be an ordered abelian group, for example, \mathbb{R} , \mathbb{Z} and \mathbb{Q} .

Definition 6.1.2. *A Λ -metric space is a pair (X, d) , where X is a set and d is a function $d : X \times X \rightarrow \Lambda$ satisfying the following conditions: for all $x, y, z \in X$, (1) $d(x, y) \geq 0$, (2) $d(x, y) = 0$ if and only if $x = y$, (3) $d(x, y) = d(y, x)$, and (4) $d(x, z) \leq d(x, y) + d(y, z)$.*

Clearly, Λ is a Λ -metric space with the metric $d(x, y) = \max(x - y, y - x)$. A closed segment in a Λ -metric space is a subset in X isometric to a closed interval $[\lambda_1, \lambda_2] = \{\lambda \in \Lambda \mid \lambda_1 \leq \lambda \leq \lambda_2\}$.

Then a Λ -tree is defined to be a Λ -metric space (T, d) satisfying the following conditions:

1. Every two points in T are the end points of a unique closed segment;
2. if two closed segments in T have a common endpoint, then their intersection is a closed segment;
3. for $a, b, c \in \Lambda$ with $a \leq c \leq b$, if $\mu : [a, b] \rightarrow T$ is a map such that $\mu|_{[a, c]}$ and $\mu|_{[b, c]}$ are parametrized segments and $\mu([a, c]) \cap \mu([c, b]) = \{\mu(c)\}$, then μ is a parametrized segment.

When $\Lambda = \mathbb{R}$, (2) is automatically satisfied [MS1, Lemma II.1.1], and (3) corresponds to the condition of containing no loops, and hence an \mathbb{R} -tree defined here is also an \mathbb{R} -tree in the sense defined earlier.

When Λ is a subgroup of \mathbb{R} , then we can take a Λ -tree as the set of vertices and fill in suitable edges to get an \mathbb{R} -tree [MS1, Theorem II.1.9].

Proposition 6.1.3. *Assume that $\Lambda \subseteq \mathbb{R}$. Then every Λ -tree T can be embedded isometrically in a unique \mathbb{R} -tree $T_{\mathbb{R}}$ such that (a) $T_{\mathbb{R}}$ is a complete metric space, (b) the union of all closed segments in $T_{\mathbb{R}}$ with endpoints in T is a dense subset of $T_{\mathbb{R}}$. If Λ is dense in \mathbb{R} , then T is dense in $T_{\mathbb{R}}$. If Λ is discrete and generated by $\ell > 0$, then $T_{\mathbb{R}}$ can be identified with a simplicial tree whose set of vertices is equal to T and whose edges have a common length ℓ .*

If Λ is a dense subset of \mathbb{R} , this proposition may give an \mathbb{R} -tree which is not a simplicial tree. To illustrate this, let $F = \overline{\mathbb{Q}_p}$ be the algebraic closure of the p -adic field, where $p \geq 2$ is a prime number. The discrete valuation of \mathbb{Q}_p extends to a non-discrete valuation of F . Let Λ be the subgroup of \mathbb{R} generated by the values of this valuation. Let \mathcal{O} be the valuation ring of F . Then there is a Λ -tree T associated to the simple algebraic group $SL(2, F)$ [MS1, II. 3] (see [Mor4] also for the higher dimensional case).

In fact, a lattice in F^2 is a finitely generated \mathcal{O} -module in F^2 which spans F^2 . Two lattices are called equivalent if one is the multiple of another by a nonzero element of F . Then points of the Λ -tree are given by equivalence classes of lattices in F^2 . To define a distance between any two such equivalence classes, we note that for any two lattices L and L' , there exists a unique lattice L'_0 which is equivalent to L' and contained in L such that L/L'_0 is isomorphic to $\mathcal{O}/\beta\mathcal{O}$ for some $\beta \in \mathcal{O}$ [MS1, Proposition II.3.5]. Then the distance between the equivalence classes $[L]$ and $[L']$ is equal to the valuation of β .

On the other hand, simplicial trees also arise from the construction in this proposition. For example, take the Λ -metric space T to be equal to Λ . Then the associated \mathbb{R} -tree is equal to \mathbb{R} .

As will be seen later, \mathbb{R} -trees also occur naturally as limit objects in topology and geometry.

6.2 Applications of \mathbb{R} -trees in topology

The original application of the theory of \mathbb{R} -trees [MS1] is to give an alternative proof of compactness of the space of hyperbolic structures of certain three dimensional manifolds in [Thu1] (see Theorem 6.2.2 below), which plays an important role in Thurston's geometrization program (Theorem 6.2.1). This theory has also been used in many other problems in topology and geometry.

Let Γ be a torsion free finitely generated group. An n -dimensional hyperbolic structure on Γ is a pair (N, φ) such that N is an n -dimensional hyperbolic manifold and $\varphi : \Gamma \rightarrow \pi_1(N)$ is an isomorphism. Each hyperbolic structure corresponds to a faithful representation $\rho : \Gamma \rightarrow SO^0(1, n)$ with discrete image.

Two hyperbolic structures (N_1, φ_1) and (N_2, φ_2) on Γ are equivalent if there is an isometry between N_1, N_2 which commutes with the isomorphisms φ_1, φ_2 .

Two representations ρ_1, ρ_2 of Γ in $SO^0(1, n)$ are equivalent if and only if they are conjugate. This implies that the set $\mathcal{H}^n(\Gamma)$ of equivalence classes of hyperbolic structures on Γ is naturally identified with the set of conjugacy classes of faithful representations of Γ in $SO^0(1, n)$ with discrete images. This identification gives a natural topology on $\mathcal{H}^n(\Gamma)$.

If Γ is the fundamental group of an n -dimensional hyperbolic manifold M of finite volume with $n \geq 3$, then the Mostow strong rigidity (Theorem 3.4.1) implies the hyperbolic structure on M is unique and $\mathcal{H}^n(\Gamma)$ consists of only one point. In particular, $\mathcal{H}^n(\Gamma)$ is compact. Hence, the compactness of $\mathcal{H}^n(\Gamma)$ is a weak version of the Mostow rigidity.

It is well-known that for $n = 2$, $\mathcal{H}^n(\Gamma)$ is noncompact. If Γ is the fundamental group of a compact hyperbolic surface, then for $n \geq 3$, $\mathcal{H}^n(\Gamma)$ contains $\mathcal{H}^2(\Gamma)$ and is noncompact. In fact, the closure of $\mathcal{H}^2(\Gamma)$ in $\mathcal{H}^n(\Gamma)$ is noncompact and hence $\mathcal{H}^n(\Gamma)$ is noncompact. The three dimensional hyperbolic manifolds in $\mathcal{H}^3(\Gamma)$ which belong to the subset $\mathcal{H}^2(\Gamma)$ have infinite volume and have natural compactifications which are manifolds with nonempty boundaries; in fact, the boundary consists of two Riemann surfaces. In general, if Γ is the fundamental group of a hyperbolic manifold of infinite volume (of finite geometric type), then $\mathcal{H}^n(\Gamma)$ might be noncompact. As explained below, manifolds with boundary occur naturally in studying three dimensional topology.

An important result of the Thurston geometrization program is the following hyperbolization Theorem of Thurston. For simplicity, we state only the version for closed manifolds (see [Mor3, pp. 51-52] [Kap, Preface] [Ot1] for general results).

Theorem 6.2.1. *If M is a compact orientable irreducible atoroidal Haken 3-manifold without boundary, then M admits a complete hyperbolic structure of finite volume.*

In this theorem, a 3-manifold M is called irreducible if every sphere $S^2 \subset M$ bounds a 3-ball in M . A closed 3-manifold M is called atoroidal if it is irreducible and every map from the torus T^2 to M has a nontrivial kernel on the fundamental group. A 3-manifold M is called Haken if it contains a properly embedded incompressible surface S not equal to the two sphere, where by incompressibility, we mean that the induced map $\pi_1(S) \rightarrow \pi_1(M)$ is injective (see [Mor3] and [Kap] for these definitions in the general case).

In proving this, the following compactness result [Thu1, Theorem 1.2] is crucial.

Theorem 6.2.2. *If Γ is the fundamental group of a three dimensional acylindrical manifold M , then $\mathcal{H}^n(\Gamma)$ is compact.*

In the above theorem, a 3-manifold M is called acylindrical if it does not contain an incompressible torus or essential annuli.

The proof of Theorem 6.2.1 is basically divided in two cases: (1) the manifold M contains an incompressible surface which is not a fiber of a fibration over the circle, (2) the manifold M is a surface bundle over the circle.

In Case (1), let $S \subset M$ be an incompressible surface that separates M into two compact manifolds M_1, M_2 with boundary. Then S is contained in the boundaries of M_1, M_2 . Denote the image of S in them by S_1 and S_2 . By induction, assume

that both M_1, M_2 have hyperbolic metrics. The problem is to deform the metrics in the space of hyperbolic metric so that the metrics in neighborhoods of S_1 and S_2 match and can be glued into a hyperbolic metric on M . Since the theory of fixed points of maps on compact spaces is used, the compactness of $\mathcal{H}^n(\pi(M_i))$, $i = 1, 2$, is crucial. See [Kap, Preface] [Mor3] for detailed outline of the arguments.

In Case (2), M is a surface bundle over S^1 and hence a mapping torus of a homeomorphism of a surface. In the proof, an important ingredient is the double limit theorem. After the original proof of Thurston, Otal [Ot2] gave an alternative proof for the generic case which depends crucially on the theory of group actions on \mathbb{R} -trees developed by Morgan-Shalen and others. To state the double limit theorem and indicate how \mathbb{R} -trees enter, we quote a part of the Review of [Ot2] by L.Mosher in Math Reviews:

“To state the double limit theorem, consider the space $\mathcal{QF}(S)$ of quasi-Fuchsian representations of $\pi_1(S)$ into $\text{Isom}(\mathbf{H}^3)$ up to conjugacy (to simplify matters, assume S is closed). A theorem of Ahlfors and Bers parameterizes the space $\mathcal{QF}(S)$ as the product of two copies of the Teichmüller space $\mathcal{T}(S)$ of S . In order to investigate what happens to a representation as the parameters in $\mathcal{T}(S)$ go to infinity, Thurston used a compactification $\bar{\mathcal{T}}(S) = \mathcal{T}(S) \cup \mathcal{PML}(S)$, where $\mathcal{PML}(S)$ is the space of projective classes of measured laminations on S . Consider a sequence $\rho_i = (\sigma_i^+, \sigma_i^-)$ in $\mathcal{QF}(S)$ such that the sequences $(\sigma_i^+), (\sigma_i^-) \in \mathcal{T}(S)$ converge, respectively, to the projective classes of two measured geodesic laminations λ^+, λ^- on S . If these two laminations fill up the surface, that is, if each component of $S - (\lambda^+ \cup \lambda^-)$ is simply connected, then the double limit theorem says that ρ_i has a subsequence converging to some representation of $\pi_1(S) \rightarrow \text{Isom}(\mathbf{H}^3)$.

”To sketch the author’s new proof of the double limit theorem, suppose ρ_i has no convergent subsequence. Applying a theorem of Morgan and Shalen, from the degenerating sequence of representations ρ_i one extracts an action of $\pi_1(S)$ on a real tree \mathcal{T} , with small arc stabilizers. By a theorem of Skora [Sk], the action of $\pi_1(S)$ on \mathcal{T} is dual to a measured geodesic lamination λ on S . The author now studies “realizations” of geodesic laminations in real trees, leading to the conclusion that (essentially) the only geodesic laminations on S which are not realizable on \mathcal{T} are those which do not cross λ . Since λ^-, λ^+ fill up S , at least one of them crosses λ , and so is realizable on \mathcal{T} . If, say, λ^- is realizable on \mathcal{T} , then for simple closed curves $\gamma \subset S$ that approximate λ^- in $\mathcal{PML}(S)$, one obtains an estimate for $l_{\rho_i}(\gamma)$, the translation length of γ with respect to the quasi-Fuchsian representation ρ_i . This is shown to contradict Ahlfors’ estimate⁹, which compares $l_{\rho_i}(\gamma)$ to $l_{\sigma_i^-}(\gamma)$, the translation length of γ with respect to the hyperbolic structure σ_i^- . ”

For more detailed outlines of the proof and other discussions, see [Ot2] and also [Kap, Chap. 18].

In a series of three papers by Morgan and Shalen [MS1] [MS2] [MS3], they gave an alternative proof of Theorem 6.2.2 using \mathbb{R} -trees and characterizations of Γ -actions on them. Since the theory works similarly for all $n \geq 3$, we discuss the

⁹It probably should be Bers instead of Ahlfors

general case of \mathbf{H}^n , $n \geq 3$, below.

The basic steps are as follows:

1. If $\mathcal{H}^n(\Gamma)$ is noncompact, then unbounded sequences will lead to some actions of Γ on \mathbb{R} -trees.
2. Show that such actions on \mathbb{R} -trees will lead to splittings of Γ . Therefore, if such splittings of Γ are excluded, the space $\mathcal{H}^n(\Gamma)$ is compact.

The \mathbb{R} -trees appear in the boundary of the Morgan-Shalen compactification of $\mathcal{H}^n(\Gamma)$, which is defined as follows. Let $\rho : \Gamma \rightarrow SO^0(1, n)$ be a faithful representation with discrete image. For each $\gamma \in \Gamma$, we define a translation length as follows: if $\rho(\gamma)$ is parabolic, define its translation length to be zero; if it is hyperbolic, $\rho(\gamma)$ fixes a unique geodesic, called the axis of translation of $\rho(\gamma)$, in \mathbf{H}^n and acts as translation on it, and define its translation length $\ell(\rho(\gamma))$ to be the translation length on the geodesic invariant under it. Let \mathcal{C} be the conjugacy classes in Γ . Then each ρ defines a translation length function

$$\ell_\rho : \mathcal{C} \rightarrow [0, +\infty)^{\mathcal{C}}, \quad \gamma \mapsto \ell(\rho(\gamma)).$$

Let $P(\mathcal{C}) = \mathbb{R}^{\mathcal{C}}/\mathbb{R}^\times$ be the projective space based on \mathcal{C} . Then ℓ_ρ defines a point in $P(\mathcal{C})$, and this induces a map

$$\Theta : \mathcal{H}^n(\Gamma) \rightarrow P(\mathcal{C}),$$

which can be shown to be an embedding. The closure of $\Theta(\mathcal{H}^n(\Gamma))$ is compact and defines the Morgan-Shalen compactification, denoted by $\overline{\mathcal{H}^n(\Gamma)}^{MS}$.

The relation of the boundary points in $\overline{\mathcal{H}^n(\Gamma)}^{MS}$ to the \mathbb{R} -tree is that if $\rho_j : \Gamma \rightarrow SO^0(1, n)$ represent a unbounded sequence of points in $\mathcal{H}^n(\Gamma)$ which converges to a point $\{\ell_\infty(\gamma)\}_{\gamma \in \mathcal{C}} \in P(\mathcal{C})$, then there exists an \mathbb{R} -tree with a Γ -action whose translation length function projects to the point $\{\ell_\infty(\gamma)\}_{\gamma \in \mathcal{C}}$ in $P(\mathcal{C})$ (see [MS1] and also [Mor2] for a summary).

The construction in [MS1] is purely algebraic and related to compactifications of affine varieties via the relative growth rates of regular functions at infinity. On the other hand, a geometric way to see how \mathbb{R} -trees with Γ -action occur in the boundary of $\mathcal{H}^n(\Gamma)$ was given in [Bes2] [Pau1], which is summarized in [Kap, Chap. 10] [Ot2, Chap. 2]. Briefly, it can be described as follows.

1. Scale down \mathbf{H}^n to get the tangent cone at infinity $T_\infty \mathbf{H}^n$, an \mathbb{R} -tree with a nontrivial Γ -action.
2. Take the Γ -invariant minimal \mathbb{R} -tree contained in $T_\infty \mathbf{H}^n$.

In the following, assume that Γ is not an elementary discrete subgroup of the isometry group of \mathbf{H}^n . Let g_1, \dots, g_m be a set of generators of Γ .

Let $d(\cdot, \cdot)$ be the distance function on \mathbf{H}^n . Define a function $D_j(x)$ on \mathbf{H}^n by

$$D_j(x) = \max\{d(x, \rho_j(g_1)), \dots, d(x, \rho_j(g_m))\}.$$

Since the functions $d(x, \rho_j(g_1)), \dots, d(x, \rho_j(g_m))$ are convex and bounded from below (note that the Γ -action is proper), the minimum value of $D_j(x)$ is achieved (see [Kap, p. 240]). Let $x_j \in \mathbf{H}^n$ be such a minimum point. Since the isometry group of \mathbf{H}^n acts transitively on \mathbf{H}^n , by conjugating ρ_j by suitable elements, we can assume that x_j is equal to a fixed basepoint x_0 in \mathbf{H}^n .

In most cases we are interested in, Γ is not virtually nilpotent, i.e., does not contain a nilpotent subgroup of finite index. Then the Margulis Lemma shows that D_j is uniformly bounded away from zero. Since ρ_j is not bounded and hence does not converge to a faithful discrete representation, D_j is not bounded either. Assume that $D_j \rightarrow +\infty$. Scale the metric d on \mathbf{H}^n to $\frac{1}{D_j}d$ and take the limit of the sequence of pointed metric spaces $(\mathbf{H}^n, \frac{1}{D_j}d, x_0)$.

Proposition 6.2.3. *As $D_j \rightarrow +\infty$, the sequence of pointed metric spaces $(\mathbf{H}^n, \frac{1}{D_j}d, x_0)$ converges to a geodesic metric space $(T_\infty\mathbf{H}^n, d_\infty, x_0)$ in the following sense: For any compact subset $K \ni x_0$ in $T_\infty\mathbf{H}^n$, there exists a compact subset $K_j \ni x_0$ in \mathbf{H}^n such that $(K_j, \frac{1}{D_j}d)$ converges to (K, d_∞) in the sense of Gromov-Hausdorff. The metric space $T_\infty\mathbf{H}^n$ is called the tangent cone at infinity. It is an \mathbb{R} -tree and branches everywhere. In $T_\infty\mathbf{H}^n$, every point is the intersection of infinitely many distinct geodesics, where a geodesic is an isometric embedding of \mathbb{R} .*

To understand the limit, we note that triangles in \mathbf{H}^n with side lengths greater than D_j with center x_0 in the metric d converge under the scaling to a Y -shaped subset with three segments coming out of the point x_0 . Changing these triangles and increasing their side lengths will produce infinitely many geodesics coming out of the point x_0 . To see that other points on these rays are also branch points, we use large triangles centered at other points. Specifically, let $\gamma(t), t \in (-\infty, +\infty)$ be a geodesic with $\gamma(0) = x_0$. Fix any $t > 0$. Let x_j, y_j, z_j be the vertices of a family of triangles whose side $\overline{x_j y_j}$ converges to the geodesic γ . Choose a point w_j on the side $\overline{x_j y_j}$ such that the sequence w_j converges to $\gamma(t)$. Then triangles in \mathbf{H}^n of long sides and with center w_j will converge to geodesics intersecting at $\gamma(t)$. This shows that the tangent cone at infinity $T_\infty\mathbf{H}^n$ is an \mathbb{R} -tree which branches infinitely many times at every point.

Remark 6.2.4. The existence of the tangent cone at infinity of \mathbf{H}^n is a special case of the tangent cone at infinity of symmetric spaces in [KL2]. In fact, according to [KL2, §5, §2.4], one first chooses a non-principal ultrafilter, which is a finitely additive probability measure ω on the collection of subsets of the natural numbers \mathbb{N} such that (1) $\omega(S) = 0$ or 1 for any $S \subset \mathbb{N}$, and (2) $\omega(S) = 0$ if S is finite. Roughly, subsets S with $\omega(S) = 1$ are suitable subsequences of \mathbb{N} , and ultralimits are basically limits for some complicated procedure of taking limits over subsequences. It is known that any sequence of pointed metric spaces has an ultralimit associated with the ultrafilter ω . In particular, every metric space (M, d) has a tangent cone at infinity, defined as the ultralimit of $(M, \frac{1}{D_j}d, x_0)$ associated with ω , where D_j is a sequence going to infinity. For symmetric spaces, it is believed by many people that the tangent cone at infinity should not depend on the choice of the ultrafilter ω , though there is no written proof of it.

Remark 6.2.5. The reason for the choice of the scaling constant D_j is that for any element in Γ , its translation length in the limit \mathbb{R} -tree is finite, and for some elements, the translation length is not zero. For this purpose, the assumption that the minimum point x_j is equal to (or conjugated to) the fixed basepoint x_0 is also important. Basically, x_0 is not far away from the axes of translation of the elements g_1, \dots, g_m . Certainly we can modify D_j by any positive number. This corresponds to the factor \mathbb{R}^\times in the projection map $\mathbb{R}^C \rightarrow P(\mathcal{C}) = \mathbb{R}^C/\mathbb{R}^\times$ above.

Consider $(\mathbf{H}^n, \frac{1}{D_j}d, x_0)$ as a pointed Γ -space through the representation ρ_j . After passing to a subsequence, it converges to the space $T_\infty\mathbf{H}^n$ with an isometric action of Γ . This finishes Step (1).

It should be emphasized that this \mathbb{R} -tree $T_\infty\mathbf{H}^n$ with the Γ -action is not necessarily the \mathbb{R} -tree which appears in the boundary of $\mathcal{H}^n(\Gamma)$. In fact, let \mathcal{T} be the minimal invariant sub- \mathbb{R} -tree in $T_\infty\mathbf{H}^n$, where by a minimal Γ -tree, we mean that it does not contain any proper Γ -invariant subtree. (see [Pau1] [Kap, Theorem 10.12, Theorem 10.24] [Ot2, pp. 28-32]). The translation lengths of the elements of Γ on \mathcal{T} are finite but not all zero and project to the limit of ρ_j in $P(\mathcal{C})$. Then \mathcal{T} is the \mathbb{R} -tree in the boundary of $\mathcal{H}^n(\Gamma)$ to which the sequence ρ_j converges.

Remark 6.2.6. There are reasons for picking the minimal Γ -invariant subtree \mathcal{T} instead of the whole tangent cone $T_\infty\mathbf{H}^n$. One reason is that minimal non-unipotent Γ -trees are uniquely determined by their translation length functions [CM] [Kap, Theorem 10.15]. Since the Morgan-Shalen compactification of $\mathcal{H}^n(\Gamma)$ is obtained by embedding into the projective space via the translation length function, it is natural to require that the boundary trees are minimal in order to get unique ones.

The \mathbb{R} -trees with Γ -action which appear in the boundary of $\mathcal{H}^n(\Gamma)$ satisfy the following important property [MS2, Theorem IV.1.2] (see also [Kap, Theorem 10.24] [Mor2, Lemma 6, Addendum 8']):

Proposition 6.2.7. *Let T be such an \mathbb{R} -tree in the boundary of $\mathcal{H}^n(\Gamma)$. Then Γ has no fixed point on T , and for every nondegenerate segment in T , its stabilizer in Γ is small, i.e., virtually abelian.*

Corollary 6.2.8. *Suppose every action of Γ on an \mathbb{R} -tree with small stabilizers of edges (nondegenerate segments) has a fixed point, then $\mathcal{H}^n(\Gamma)$ is compact.*

The problem is to find conditions under which the Γ -action in the corollary has no fixed point. This is given as follows [MS2, Chap. IV] [Mor2, Theorem 12].

Proposition 6.2.9. *Let Γ be a finitely generated group isomorphic to the fundamental group of a 3-manifold. If Γ has an action without fixed points on an \mathbb{R} -tree with small edge stabilizers, then Γ splits as either (I) $\Gamma = A \star_C B$ with C virtually abelian and $C \neq A$, $C \neq B$, or (II) $\Gamma = A \star_C$ with C virtually abelian.*

The compactness result of Thurston in Theorem 6.2.2 follows from this result and Corollary 6.2.8.

Remark 6.2.10. There are alternative approaches to issues related to the compactness of $\mathcal{H}^n(\Gamma)$. See [BF] [GLP] [Chi]. Characterizations of certain nice actions of surface groups on \mathbb{R} -trees are given in [Sk] and used crucially in [Ot2]. For other applications of \mathbb{R} -trees to topology, see [KIS] [AS] [Wol1] [DDW].

6.3 \mathbb{R} -Euclidean buildings

In §6.1, we recalled the definition of \mathbb{R} -trees. They are special cases of \mathbb{R} -Euclidean buildings, which will be recalled briefly.

As seen above, an \mathbb{R} -tree is not a 1-dimensional simplicial complex. It contains infinitely many geodesics, which correspond to apartments in rank-1 Euclidean buildings. In general \mathbb{R} -trees, geodesics are also called apartments, but they can not be decomposed into a locally finite disjoint union of chambers and chamber faces as in the case of simplicial trees.

Roughly, an \mathbb{R} -Euclidean building of rank $r \geq 1$ is a metric space which contains many compatible apartments, each of them is an affine (or Euclidean) space of dimension n admitting the action of a generalized Weyl group.

More precisely, let \overline{W} be a finite Coxeter group acting essentially on a vector space \mathbb{R}^r . Let \mathbf{A} be the affine space associated with \mathbb{R}^r . Define W to be the group of affine isometries of \mathbf{A} whose vector part is \overline{W} , i.e., $W = \overline{W} \cdot \mathbb{R}^r$. We can also take a subgroup of W such that its vector part is equal to \overline{W} but the translation part is a dense subgroup of \mathbb{R}^r . Using reflections in W , we can define half-apartments, sectors and sector faces as in the case of Euclidean reflection complexes. (It should be emphasized that we could not define chambers since the walls, i.e., the faces of half apartments, are dense in the apartments).

Then an \mathbb{R} -Euclidean building modeled on (\mathbf{A}, W) is a metric space Δ together with a family \mathcal{F} of isometric embeddings of \mathbf{A} into Δ , which are called the apartments in Δ , satisfying the following conditions:

1. If $w \in W$ and $f \in \mathcal{F}$, then $f \circ w \in \mathcal{F}$.
2. If $f, f' \in \mathcal{F}$, then $S = f^{-1}(f'(\mathbf{A}))$ is closed and convex in \mathbf{A} (this should be interpreted as the intersection of the two apartments $f(\mathbf{A})$ and $f'(\mathbf{A})$), and there exists $w \in W$ such that $f|_S = f' \circ w|_S$.
3. Any two points in Δ lie in a common apartment.
4. Any two sectors contain subsectors which are contained in a common apartment.
5. If A_1, A_2, A_3 are three apartments such that each of $A_1 \cap A_2$, $A_1 \cap A_3$ and $A_2 \cap A_3$ is a half-apartment, then $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

Remark 6.3.1. We basically follow the definition given in [Ro3, Appendix 3]. Conditions (1)-(4) are reasonable and clearly expected, and (5) can be replaced by the following: For any apartment $f(\mathbf{A})$ and a point p in it, there exists a retraction $\rho : \Delta \rightarrow f(\mathbf{A})$ such that $\rho^{-1}(p) = p$, and the restriction to each apartment is distance decreasing.

Remark 6.3.2. There are other definitions of \mathbb{R} -Euclidean buildings or affine \mathbb{R} -buildings (and more generally affine Λ -buildings). The first non-discrete buildings were introduced by Bruhat and Tits in [BT1, §7]. The general axioms of non-discrete buildings were first introduced by Tits in [Ti7]. In [KL2, 4.1.2], \mathbb{R} -Euclidean buildings were defined in a slightly different way. In [Par], Parreau showed that they are equivalent. See also the survey [KT].

As recalled earlier in §4.2, for any field F with a non-discrete valuation $\nu : F^\times \rightarrow \mathbb{R}_{\geq 0}$, there is a natural \mathbb{R} -tree associated with $SL(2, F)$. In [BT1, §7.4], for a linear connected and simply connected semisimple algebraic group \mathbf{G} defined over such a field F , there is an \mathbb{R} -Euclidean building $\Delta^{BT}(\mathbf{G})$. When the valuation is discrete, the building is a simplicial complex and can be described by either specifying the set of simplexes, for example, the vertices and chambers; or using a BN -pair. In the case of rank 1, i.e., Λ -trees, we specified the set of vertices and described the edges using a Λ -distance. This approach seems to be difficult to generalize in the higher rank case. Hence, Bruhat and Tits [BT1, §7] used another construction which also gives an alternative construction of the usual Euclidean buildings.

To motivate this construction, we consider the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ of a simply connected algebraic group over a local field $k_{\mathfrak{p}}$. Then $\mathbf{G}(k_{\mathfrak{p}})$ acts transitively on the set of apartments. Fix an apartment Σ , which can be identified with an affine space \mathbf{A} . Then there is a surjective map

$$\mathbf{G}(k_{\mathfrak{p}}) \times \Sigma \rightarrow \Delta^{BT}(\mathbf{G}).$$

The fibers of this map can be described explicitly in terms of parabolic subgroups.

The construction for the non-discrete case is a generalization of this (see [Ti3, §2] and also [Lan2]). Let \mathbf{T} be a maximal split torus in \mathbf{G} , and N the normalizer of $\mathbf{T}(F)$ in $\mathbf{G}(F)$. Then the group $X_*(\mathbf{T}) = \text{Hom}_F(GL_1, \mathbf{T})$ determines a real vector space $V = X_*(\mathbf{T}) \otimes \mathbb{R}$. Let $Z = Z(\mathbf{T}(F))$ be the centralizer of $\mathbf{T}(F)$ in $\mathbf{G}(F)$. There is a natural homomorphism $\nu : Z \rightarrow V$. Denote its kernel by Z_c . Then $\Lambda = Z/Z_c$ is a free abelian group, and $W = N/Z_c$ is an extension of a finite Coxeter group by Λ , and acts isometrically on the affine space \mathbf{A} associated with V (see [Ti3, §1.2]).

In the affine space \mathbf{A} , we can define various subgroups associated to subsets, in particular to points, for example, \hat{P}_x , $x \in \mathbf{A}$ [BT1, §7.1]. Then the associated building $\Delta^{BT}(\mathbf{G})$ is defined as a quotient of $\mathbf{G}(F) \times \mathbf{A}$ under the following equivalence relation: Two pairs (g, x) and (h, y) are equivalent if there exists $n \in N$ such that

$$y = n \cdot x, \quad \text{and} \quad g^{-1}hn \in \hat{P}_x.$$

6.4 Quasi-isometry rigidity and tangent cones at infinity of symmetric spaces

As seen in the proof of Mostow strong rigidity in Theorem 3.4.1, quasi-isometries play an important role. In fact, a crucial step is that under the equivariant quasi-isometry φ in Equation 3.4.2, the image of a flat F in X lies within a bounded neighborhood of a flat F' in X' (Proposition 3.4.4).

Gromov initiated a systematic study of global geometry of groups in terms quasi-isometries (see [Gr]). One problem concerns the rigidity of symmetric spaces of higher rank under quasi-isometries, a conjecture due to Margulis in late 1970's, which was proved by Kleiner and Leeb [KL2, Theorem 1.1.3] and later Eskin and Farb [EF1]. One result in the proof says that in the above result of Mostow in Proposition 3.4.4, the equivariance condition on the quasi-isometry can be removed [KL2, Corollary 7.1.5].

Proposition 6.4.1. *Let X, X' be two symmetric spaces of noncompact type without rank one irreducible factors. If they are quasi-isometric, then they are isometric after the metrics on the irreducible factors are suitably scaled, and any quasi-isometry between them is within a bounded distance of an isometry.*

Remark 6.4.2. See [KL1] [Esk] [EF1] [Scr1] [Scr2] [FS] and the review of [Esk] by L.Mosher for a summary of other results related to classifications of lattices in semisimple Lie groups up to quasi-isometry. Parts of the proofs in [Esk] follow the proof of the Mostow strong rigidity and use the spherical Tits buildings in a substantial way.

The results in [KL2, Theorem 1.1.3] are more general than stated here. The proof in [KL2] of this result goes as follows:

1. Choose an ultrafilter ω and obtain the tangent cones at infinity $T_\infty X$ and $T_\infty X'$.
2. Show that $T_\infty X$ and $T_\infty X'$ are \mathbb{R} -Euclidean buildings of rank at least two, and a quasi-isometry $\varphi : X \rightarrow X'$ induces a homeomorphism between the two \mathbb{R} -Euclidean buildings.
3. The homeomorphism maps an apartment in $T_\infty X$ to an apartment in $T_\infty X'$.
4. Let F be a flat in X . Then $\varphi(F)$ lies in a bounded distance of a flat F' of X' .
5. Follow similar steps in the proof of Mostow strong rigidity to prove the result.

For a more detailed outline and summary of the proof, see [KL2, §1.2].

Remark 6.4.3. In the proof of the Mostow strong rigidity, we push things out to infinity to get an isomorphism of the spherical Tits buildings. In Step (1) of the proof in [KL2] outlined here, we pull in infinity towards a basepoint to get the tangent cones at infinity, and the symmetric spaces become \mathbb{R} -Euclidean buildings. Both have the common feature that they ignore finite ambiguities and turn a quasi-isometry into more precise maps.

7 Twin buildings and Kac-Moody groups

In this section, we briefly introduce twin buildings, Kac-Moody groups and some geometric group theoretical properties of Kac-Moody groups over finite fields. Then we also mention other groups in geometric group theory that arise from actions on buildings and related spaces.

7.1 Twin buildings

Roughly speaking, a twin building is a pair of buildings of the same type (i.e., the same Coxeter system for their apartments) with a notion of *codistance* between every pair of chambers from this pair of buildings.

More precisely, we follow [AbB, Definition 5.133] and describe buildings as W -metric spaces.

Definition 7.1.1. Let (W, S) be a Coxeter system, i.e., W be a Coxeter group and S be a minimal set of generators consisting of elements of order 2. A twin building of type (W, S) is a triple $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ consisting of two buildings $(\mathcal{C}_+, \delta_+)$ and $(\mathcal{C}_-, \delta_-)$ of type (W, S) together with a codistance function

$$\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \rightarrow W$$

satisfying the following conditions for each $\varepsilon \in \{+, -\}$, any $C \in \mathcal{C}_\varepsilon$, and any $D \in \mathcal{C}_{-\varepsilon}$, where $w = \delta^*(C, D)$:

1. $\delta^*(C, D) = \delta^*(D, C)^{-1}$.
2. If $C' \in \mathcal{C}_\varepsilon$ satisfies $\delta_\varepsilon(C', C) = s$ with $s \in S$, and $\ell(sw) < \ell(w)$, then $\delta^*(C', D) = sw$.
3. For any $s \in S$, there exists a chamber $C' \in \mathcal{C}_\varepsilon$ with $\delta_\varepsilon(C', C) = s$ and $\delta^*(C', D) = sw$.

An effective way to construct a Tits building is to use BN -pairs. Similarly, an effective way to construct a twin building is to use a twin BN -pairs [AbB, §6.3.3].

Definition 7.1.2. Let B_+ , B_- and N be subgroups of a group G such that $B_+ \cap N = B_- \cap N$, which is denoted by T . Assume that T is a normal subgroup of N , and set $W = N/T$. The triple (B_+, B_-, N) is called a *twin BN -pair* with Weyl group W if W admits a set of generators such that the following conditions hold for all $w \in W$ and $s \in S$ and each $\varepsilon \in \{+, -\}$:

1. (G, B_ε, N, S) is a Tits system.
2. If $\ell(sw) < \ell(w)$, then $B_\varepsilon s B_\varepsilon w B_{-\varepsilon} = B_\varepsilon s w B_{-\varepsilon}$.
3. $B_+ s \cap B_- = \emptyset$.

For any twin Tits system (G, B_+, B_-, N, S) with the Weyl group $W = N/T$, there is a twin building $(\mathcal{C}_+, \mathcal{C}_-)$ with $\mathcal{C}_+ = G/B_+$, $\mathcal{C}_- = G/B_-$ and the codistance function δ^* given by

$$\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w \text{ if and only if } g^{-1}h \in B_\varepsilon w B_{-\varepsilon}$$

for $g, h \in G, w \in W$, and $\varepsilon \in \{+, -\}$. Note that we have used an analogue of the Bruhat decomposition, $G = B_- W B_+$ [AbB, Proposition 6.75].

Twin buildings constructed from twin B -pairs have the property that G acts strongly transitively on the twin building, i.e., G acts transitively on the set

$\{(C, D) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta^*(C, D) = w\}$ for every $w \in W$. Conversely, given any twin building, if the automorphism group acts strongly transitively, then it admits a twin Tits system, and the twin building is the same as the building constructed from the twin Tits system [AbB, Theorem 6.87].

An effective method to construct a twin Tits system is to construct a group root datum, called a *RGD-system*. See [Abr1, pp. 14-15] [CaR3].

7.2 Kac-Moody algebras and Kac-Moody groups

A finite dimensional complex Lie algebra is determined by its Cartan matrix.

Briefly, a *Cartan matrix* is an integer valued matrix $A = (A_{ij})$, $i, j = 1, \dots, r$, satisfying the following conditions:

1. $A_{ii} = 2$.
2. For $i \neq j$, $A_{ij} \leq 0$.
3. $A_{ij} = 0$ if and only if $A_{ji} = 0$.
4. There exists a diagonal matrix D such that $D^{-1}A$ is symmetric and positive definite.

Given any finite dimensional simple complex Lie algebra \mathfrak{g} , there is a Cartan matrix associated with it by choosing a set of simple roots $\alpha_1, \dots, \alpha_r$, where r is the rank of \mathfrak{g} , as follows:

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the Killing form.

On the other hand, given a Cartan matrix, we can construct a finite dimensional complex Lie algebra by using the Serre relations.

In this correspondence, a complex Lie algebra is simple if and only if the Cartan matrix is irreducible, i.e., there does not exist a proper subset $I \subset \{1, \dots, r\}$ such that $A_{ij} = 0$ for $i \in I, j \notin I$.

A *generalized Cartan matrix* is an integer valued matrix $A = (A_{ij})$ satisfying all the above conditions for a Cartan matrix except for (4) on the positivity of $D^{-1}A$.

Given a generalized Cartan matrix, there is a Kac-Moody algebra using Serre relations (see [Kac] for details).

For any finite dimensional simple complex Lie algebra Γ , there is a Chevalley group \mathbf{G} defined over any field k [Che] (see [Car2] for a detailed exposition). Similarly, for a Kac-Moody algebra, there is also a Kac-Moody group \mathcal{G} over k [Ti9]. For simplicity, we call the group of k -points of \mathcal{G} also the Kac-Moody group and denote it by $\mathcal{G}(k)$.

An important class of Kac-Moody groups consists of affine Kac-Moody groups such as $SL(n, k((t)))$, where k is any field, and t is variable and $k((t))$ is the field of formal power series, or more generally $\mathbf{G}(k((t)))$, where \mathbf{G} is a Chevalley group.

The important point here is that for any Kac-Moody group \mathcal{G} , there are natural choices of twin Tits systems and hence twin buildings $(\mathcal{C}_+, \mathcal{C}_-)$. Briefly, fix a

choice of positive roots, and take B_+ as the subgroup generated by all positive root groups, and B_- as the subgroup generated by all negative roots. With a suitable choice of N , they give rise to a twin Tits system (G, B_+, B_-, N, S) . See [Abr1, Proposition 1, p. 16] [Ro4, p. 181] for more details and a precise definition of N .

Recall that $\mathcal{C}_+, \mathcal{C}_-$ are buildings defined as W -metric spaces. Denote their corresponding canonical simplex buildings by Δ_+, Δ_- . The product space $\Delta_+ \times \Delta_-$ is also called a twin building associated with the Kac-Moody group.

For more details of the results stated below, see [Re8] and references there.

Proposition 7.2.1. *If $\mathcal{G}(k)$ is a Kac-Moody group defined over a finite field k , then it is a finitely generated group and acts properly on the twin building $\Delta_+ \times \Delta_-$ with finite stabilizers.*

When the field k is finite, the buildings Δ_+, Δ_- are locally finite, and their automorphism groups $\text{Aut}(\Delta_+) \times \text{Aut}(\Delta_-)$ are locally compact subgroups and hence admit Haar measures.

Proposition 7.2.2. *Assume $\mathcal{G}(k)$ is a Kac-Moody group over a finite field $k = F_q$ of q elements and that the Weyl group W of \mathcal{G} is infinite. Denote by $W(t) = \sum_{w \in W} t^{\ell(w)}$ its growth function of W . If $W(1/q) < +\infty$, then $\mathcal{G}(k)$ is a lattice of the locally compact group $\text{Aut}(\Delta_+) \times \text{Aut}(\Delta_-)$, and the value $W(1/q)$ is the volume of $\mathcal{G}(k) \backslash \text{Aut}(\Delta_+) \times \text{Aut}(\Delta_-)$ with respect to a natural normalization of the Haar measure.*

When W is an affine Weyl group, it grows polynomially and hence the condition $W(1/q) < +\infty$ is automatically satisfied since $1/q < 1$.

Affine Kac-Moody groups over finite fields such as $SL(n, F_q[t, t^{-1}])$ are S -arithmetic subgroups of linear algebraic groups defined over function fields. They admit many finite index normal subgroups such as congruence subgroups in the same way arithmetic subgroups such as $SL(n, \mathbb{Z})$ admit many finite index normal subgroups.

An important result of [CaR1] is that among Kac-Moody groups, this is essentially the only case where it can happen.

Proposition 7.2.3. *Let $\mathcal{G}(k)$ be a Kac-Moody group defined over a finite field F_q . Assume that the Weyl group W is infinite and irreducible and $W(1/q) < +\infty$, and the buildings Δ_{\pm} are not Euclidean and $\mathcal{G}(k)$ is generated by its root subgroups. Then \mathcal{G} modulo its finite center is a simple group.*

This result produces many finitely generated infinite simple groups. Before this work, there was another family of finitely presented infinite simple groups which are lattices in the automorphism groups of product of trees in [BuM1].

The above proposition is consistent with nonlinearity result [Re6, Theorem 5.4.4].

Proposition 7.2.4. *Let $\mathcal{G}(F_q)$ be a Kac-Moody group whose twin buildings are associated with a right-angled Fuchsian group as the Coxeter group. Assume that any prenilpotent pair of roots not contained in a spherical root system leads to a*

trivial commutation of the corresponding root groups. Then for $q \gg 1$, the group $\mathcal{G}(F_q)$ is not linear over any field.

Remark 7.2.5. Subgroups of Kac-Moody groups $\mathcal{G}(k)$ can also produce lattices in $Aut(\Delta_+)$ and $Aut(\Delta_-)$. In fact, as mentioned before, \mathcal{G} contains a twin Tits system $(\mathcal{G}(k), B_+, B_-, N, S)$. It turns out that when the finite field k is sufficiently large, B_+ acts properly on Δ_- as a lattice, i.e., B_+ is a lattice of $Aut(\Delta_-)$, and B_- also acts on Δ_+ as a lattice [CaG1] [Re4] [Re6].

Remark 7.2.6. The above proposition is an important example to show that the building $\Delta_+ \times \Delta_-$ can be used to construct a natural ambient *locally compact group* $Aut(\Delta_+ \times \Delta_-)$ which contains the Kac-Moody group $\mathcal{G}(k)$. We also note that B_+ is the stabilizer of a chamber in Δ_+ , and B_- is also the stabilizer of a chamber in Δ_- .

Remark 7.2.7. The completions of \mathcal{G} in the group $Aut(\Delta_\pm)$ are called complete Kac-Moody groups. For conditions on simplicity of the complete Kac-Moody groups, see [CaER].

For more discussions and introductions to Kac-Moody groups as lattices and other properties, see [Re1] [Re2] [Re6] [Re7] [Re8].

7.3 Kac-Moody groups as lattices and groups arising from buildings in geometric group theory

Besides infinite simple groups coming from Kac-Moody groups in Proposition 7.2.3, buildings and related spaces such as trees, CAT(0)-simplicial complexes and Davis complexes can be used to produce many other interesting groups in geometric group theory. In this subsection, we mention briefly several results.

In [DrJ], the following results on Coxeter groups were proved.

Proposition 7.3.1. *Every Coxeter group acts amenably on a compact space and satisfies Property A.*

As a corollary, it follows

Corollary 7.3.2. *Every Coxeter group satisfies the Novikov conjecture in C^* -algebras and hence the usual Novikov conjecture on homotopy invariance of higher signatures.*

In [DrJ], it was also proved that every Coxeter group has finite asymptotic dimension. Using the results in [Da1], we can see that every torsion-free finite index subgroup Γ admits a finite classifying space $B\Gamma$, or equivalently a cofinite $E\Gamma$ space, which can be realized as a subspace of the Davis complex. Combining them together and applying known criterions for integral Novikov conjectures (see [J4] for example for the formulation of integral Novikov conjecture for groups containing torsion elements), we observe

Proposition 7.3.3. *Every Coxeter group satisfies the integral Novikov conjecture.*

In [CaL], for any noncompact building, a combinatorial boundary was constructed where the action of $Aut(X)$ on X extends. Assume that X is locally finite, then the union of X and the combinatorial boundary gives the *combinatorial compactification* of X .

In [Lec], the following result was shown

Proposition 7.3.4. *For any locally finite building X and any locally compact subgroup G of $Aut(X)$ that acts properly on X , the action of G on the combinatorial compactification is topologically amenable.*

Corollary 7.3.5. *Let X be any locally finite building. Then any discrete subgroup of $Aut(X)$ satisfies the Novikov conjecture in C^* -algebras and hence the usual Novikov conjecture. In particular, Kac-Moody groups over finite fields satisfy the Novikov conjecture in C^* -algebras.*

Remark 7.3.6. Another method to prove the Novikov conjecture in C^* -algebras for a finitely generated subgroup Γ of $Aut(X)$ of a locally finite building X is to use the following two facts:

1. By [DyS], X has finite asymptotic dimension.
2. Any finitely generated group Γ acting properly and isometrically on X also has finite asymptotic dimension, and hence also satisfies Property A and the Novikov conjecture in C^* -algebras.

Remark 7.3.7. A natural question is whether any Kac-Moody group over a finite field satisfies the integral Novikov conjecture as formulated in [J4]. Briefly, for a torsion-free group, the integral Novikov conjecture in each theory (such as L -theory (or surgery theory), algebraic K -theory) asserts the injectivity of the assembly map in that theory. (Note that the usual Novikov conjecture for homotopy invariance of higher signatures corresponds to the injectivity of the rational assembly map in the L -theory, which is the tensor product of the assembly map with the field of rational numbers. For groups containing torsion elements, there is an assembly map for the family of finite subgroups, and the injectivity of the corresponding assembly map is called the *integral Novikov conjecture*. Since split Kac-Moody groups over finite fields are not virtually torsion-free, i.e., do not admit finite index torsion-free subgroups, this formulation with respect to the family of finite subgroups is necessary.

Groups acting on Euclidean buildings often enjoy some special properties. For example, the space of radial functions on finitely generated groups acting simply transitively on Euclidean buildings of type \tilde{A}_n satisfies the Haagerup inequality [Va] [Sw]. The same result holds for more general lattices acting on buildings [Pe]. For explicit constructions of groups acting simply transitively on Euclidean buildings of type \tilde{A}_n , see [Cart1] [CMSZ1] [CMSZ2]. Such groups are also important in the problem of determining fake projective planes.

Zeta functions of certain classes of infinite groups are computed by using enumeration in Bruhat-Tits buildings [Vo].

A special and important example of Euclidean buildings are (simplicial) trees. It is well-known that simplicial trees are fundamental in combinatorial group theory (see [Se2]). For a survey of more recent developments and generalizations, see the book [BL] and [BuM1] [BuM2]. Many natural discrete subgroups in the automorphism group of trees have been constructed from Kac-Moody groups and subgroups. In these studies, the buildings associated to the infinite dimensional Kac-Moody groups (or algebras) are used. As pointed out in the introduction, many such buildings are hyperbolic, i.e., the apartments have negative curvature. This seems to be an active ongoing research area. See [CaG1] [CaG2] [Re1] [Re4] [DJ] [Ja] [Laf1] [Laf2].

Buildings are also used by Deligne in [Del] to compute the homology groups of hyperplane arrangements.

8 Other applications of buildings

In the previous sections, we have discussed applications of buildings to many problems in geometry, geometric topology, algebraic topology, representation theory, harmonic analysis etc. In this last section, we indicate very briefly other applications.

8.1 Applications in algebraic geometry

Algebraic groups and their parabolic subgroups occur naturally in geometric invariant theory and torus embeddings (or toric varieties) and toroidal compactifications of Hermitian locally symmetric spaces. Because of this, spherical Tits buildings have played an important role in these subjects.

Given a reductive algebraic group \mathbf{G} and a variety X where \mathbf{G} -acts, a basic problem in geometric invariant theory is to find geometric quotients of suitable open subsets U of X which are invariant under \mathbf{G} and the geometric quotient of U by \mathbf{G} exists. These open subsets U can be defined in terms of notions of stability and semi-stability of points [Mu3, Chap. I, §4]. Now these stability conditions are related to the positivity of certain numerical invariants of 1-parameter subgroups λ in \mathbf{G} [Mu3, Chap. II, §1, Theorem 2.1]. Considerations of these invariants naturally lead to parabolic subgroups $\mathbf{P}(\lambda)$ of λ [Mu3, Chap. II, §2]. This connection with parabolic subgroups gives an equivalence relation on the set of 1-parameter subgroups of \mathbf{G} , which leads to a rational flag complex [Mu3, Chap. II, Definition 2.5]. The rational flag complex is contained in the spherical Tits building of \mathbf{G} as a dense subset and its completion gives the spherical Tits building $\Delta(\mathbf{G})$ [Mu3, Chap. II, pp. 59-61]. This embedding in the building $\Delta(\mathbf{G})$ allows one to define semi-convex subsets and convex subsets of the rational flag complex of 1-parameter subgroups [Mu3, Chap. II, Definition 2.10], which are important in questions about stability [Mu3, Chap. II, §3].

The 1-parameter subgroups λ of \mathbf{G} in [Mu3] describe the geometry at infinity of \mathbf{G} . A more explicit relation between the structure at infinity of \mathbf{G} and the spherical Tits building $\Delta(\mathbf{G})$ is given in [KKMS, Chap. IV, §2]. In fact, a toroidal embedding (or a partial compactification) of a semisimple linear algebraic group \mathbf{G}

over an algebraically closed field k was constructed. The boundary is decomposed into boundary components whose normalizers are exactly the parabolic subgroups of \mathbf{G} . The inclusion relations between the closures of the boundary components define a complex which turns out to be exactly the spherical Tits building $\Delta(\mathbf{G})$. One reason is that the polyhedral cone decomposition for the torus embeddings involved is exactly the Weyl chamber decomposition [KKMS, p. 185, p. 187].

Simplicial complexes constructed from only a *proper* subset of the set of all parabolic subgroups are also important in toroidal compactifications of Hermitian locally symmetric spaces [Mu1] [AMRT] [HKW] [HZ1] [HZ2]. In fact, the theories in [KKMS] were used in the sequel [AMRT] to explicitly resolve the singularities of the Baily-Borel compactification [BB] of Hermitian locally symmetric spaces, or quotients of bounded symmetric domains by arithmetic quotients. The resolutions are given by the toroidal compactifications. The reason why the Tits buildings or related complexes come in is the same basically as before. The normalizers of boundary components of (partial) compactifications of groups and bounded symmetric domains are often certain parabolic subgroups, and the inclusion relation between their closures lead to simplicial complexes which are often closely related to spherical Tits buildings. For example, in the partial Baily-Borel compactification of a bounded symmetric domain, only maximal parabolic subgroups arise as the normalizer of the boundary components. See [HKW] for detailed discussions about the (partial) Tits buildings and relations to the Baily-Borel compactification, and the toroidal compactifications of the moduli space of abelian surfaces. The discussions here are related to the point of view in §2.12.

The spherical Tits buildings of dimension 1 are crucial in the important paper [Mu2] on uniformization of curves over complete local rings by Schottky groups (see [FvP] for more recent results and applications and also [GvP] [vP], where trees and currents on trees were used), and the generalization in [Mus]. Related results about Arakelov intersection of arithmetic varieties are given in [Man], which motivated [Wer3], where the Bruhat-Tits building of $PGL(n)$ was used to compute certain Arakelov intersection. As pointed out earlier, Euclidean buildings of dimension 1 are trees. Certain trees, called valuation trees in [FJ], are important in describing the singularities in complex analysis and algebraic geometry.

The theory of Bruhat-Tits buildings plays an important role in [Ra3] on work related to a conjecture by Grothendieck and Serre that given any principal bundle over a smooth connected k -scheme with fiber given by a smooth reductive group scheme over k , if it is trivial over some nonempty open subset of X , then it is locally trivial in the Zariski topology. The Bruhat-Tits theory is also used in a new proof in [GaR] of some results of Bott on the loop space of a compact, simply connected, simple Lie group and hence of the Bott periodicity for the unitary group.

Buildings also occur naturally in rigid analytic geometry over local non-Archimedean fields. An important rigid analytic space is the p -adic symmetric space \mathcal{X}_n associated with the group $PGL(n, K)$, introduced by Drinfeld in [Dr] (see also [BoC] and [vPV]). There is a well-defined surjective equivariant map from \mathcal{X}_n to the Bruhat-Tits building of $PGL(n, K)$, called the reduction map [Te]. This reduction map plays an important role in computing the cohomology of \mathcal{X}_n and its quotients by

cocompact discrete groups in [SS2] (see also [Te]).

A natural boundary of \mathcal{X}_n is the topological spherical Tits building of $PGL(n, K)$. As recalled in §3.5, this topological Tits building appears naturally as the boundary of the Bruhat-Tits building of $PGL(n, K)$. An important property of this boundary of \mathcal{X}_n is that the reduction map extends to an equivariant map to the boundaries. This plays an important role in [ST], which generalizes the Poisson integral transformation of Riemannian symmetric spaces.

Buildings are also related to p -adic analytic spaces in the sense of [Ber2]. In fact, rigid analytic spaces are analog over local non-Archimedean fields of complex analytic spaces but do not have the corresponding topological properties, for example, local compactness and local arcwise connectedness. To overcome this problem, p -adic analytic spaces were introduced in [Ber2]. Roughly, to a rigid analytic space X , there is a p -adic analytic space X^{an} , which is a kind of completion (or filling in holes) in a suitable way.

The Bruhat-Tits buildings $\Delta^{BT}(\mathbf{G})$ can be embedded into a suitable Berkovich analytic space \mathbf{G}^{an} associated with the algebraic group \mathbf{G} and this has been used to construct compactifications of $\Delta^{BT}(\mathbf{G})$ in [ReTW1] [ReTW2].

Buildings and trees are related to these spaces in several ways. In fact, one dimensional quasipolyhedrons were introduced in [Ber2, Definition 4.1]. The simply connected ones are certain trees. If X is a smooth geometrically connected projective curve of genus $g \geq 1$, then X^{an} is a quasipolyhedron whose Betti number is at most g [Ber2, Theorem 4.3.2]. When \mathbf{G} is a semisimple Chevalley group over a local non-Archimedean field, then its Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ can be embedded into \mathbf{G}^{an} in two ways [Ber2, Theorems 5.4.2 and 5.4.4]. This is important in understanding the homotopy type of some p -adic analytic spaces [Ber2, Chap. 6]. When \mathbf{P} is a parabolic subgroup of \mathbf{G} , then the Bruhat-Tits building $\Delta^{BT}(\mathbf{G})$ can also be embedded into $\mathbf{G}^{an}/\mathbf{P}^{an}$. Since the latter is compact, the closure of $\Delta^{BT}(\mathbf{G})$ under this embedding gives a compactification of $\Delta^{BT}(\mathbf{G})$ [Ber2, Remarks 5.5.2].

The results mentioned above are related to the reductive group $PGL(n, K)$. See [RaZ] for generalizations of some of these results to some other reductive groups over local non-Archimedean fields.

8.2 Random walks and the Martin boundary

Random walks on graphs and in particular on trees are natural problems. They are related to harmonic functions, which are functions on the set of vertices and satisfy the condition that the value at any point is equal to the average (or weighted average) of values over nearby points. Closely related problems are to determine the Poisson boundary and the Martin compactification. These problems have been solved for trees [An] [Woe].

For Euclidean buildings of rank at least 2, we can also define functions on the set of vertices, or the set of simplexes of any fixed dimension, and harmonic functions. The related problems on the Poisson boundary and the Martin compactification have been only partially understood. See [CaW], [Ge], and the papers and references in [Ko].

8.3 Finite groups

Spherical Tits buildings or rather BN -pairs have played an important role in understanding finite groups and their representations [Car1] [Car2] [So]. For example, the book [TW] on Moufang polygons, i.e., Moufang spherical buildings of rank 2, is in the list of background references of the book [AbS].

A particularly important application is a criterion for simplicity of groups with BN -pairs [Ti8]. We note that this criterion for simplicity applies to both finite and infinite groups. See [Car1] [Su1] [Su2] [Su3] [Har] for characterization and classification of finite simple groups of Lie type.

The notion of BN -pairs was motivated by the fundamental work of finite groups of Lie type [Che].

Tits buildings are important for representation theories of finite groups. For example, by the Solomon-Tits Theorem, a spherical Tits building is homotopic to a wedge of sphere and hence has nonzero reduced homology in a unique degree, which gives the Steinberg module. There are many other applications to characters and representations of finite groups. See [Car2] [Lus].

8.4 Finite geometry

Buildings are also important in incidence geometry (see the many articles in [Bue]), finite geometry (see [Hir] [Tha2] and the references there) and coding theory [MS].

A finite geometry is any geometric system that has only a finite number of points. It can be defined by linear algebra as vector spaces and related structures such as projective spaces over a finite field, which is called Galois geometry, or can be defined purely combinatorially. For example, a projective space S can be defined abstractly as a set P (the set of points), together with a set L of subsets of P (the set of lines), satisfying these axioms:

1. Every pair of two distinct points are in exactly one line.
2. Veblen's axiom: If a, b, c, d are distinct points and the lines through ab and cd meet, then so do the lines through ac and bd .
3. Any line has at least 3 points on it.

This is closely related to incidence geometry. For example, an incidence geometry of dimension 2 is a geometry satisfying the conditions:

1. Every pair of distinct points determines a unique line.
2. Every line contains at least two distinct points.
3. For every line, there is at least one point that does not lie on the line.

Finite Tits buildings give finite geometries. Many finite geometries are Galois geometries, for example, any finite projective space of dimension three or greater is isomorphic to a projective space over a finite field, but in dimension two there are combinatorially defined projective planes which are not isomorphic to projective spaces over finite fields, namely the non-Desarguesian planes, which are projective

planes that do not satisfy Desargues's theorem: *In a projective space, two triangles are in perspective axially if and only if they are in perspective centrally.*

8.5 Algebraic K -groups

An important conjecture of Bass in algebraic K -theory concerns finite generation of K -groups associated with a commutative ring which is finitely generated as an algebra over \mathbb{Z} (see [FW, p. 53] for the precise statement). This conjecture has been proved by Quillen for finite fields [Qu2], the ring of integers of any number field [Qu1] and the ring of regular functions of a smooth affine curve over a finite field (written up by Grayson in [Gra3]. See also [Ro2, §13]).

In [Qu], the Solomon-Tits theorem that the homotopy type of a spherical Tits building is a bouquet of spheres was used crucially. In [Gra3], Bruhat-Tits buildings of $GL(n)$ over non-archimedean local fields induced from function fields played an important role.

As pointed out in [Qu1], the finiteness result also holds for the ring of S -integers \mathcal{O}_S in any number field, by combining the results for finite fields and a localization result. The result in [Gra3] can also be restated as that the K -groups of rings of S -integers (see §3.6 for definition) of the function field of a smooth projective curve over a finite field are finitely generated.

8.6 Algebraic combinatorics

Complexes similar to Coxeter complexes and spherical Tits buildings also occur naturally in algebraic combinatorics. In fact, they are very important special examples of these general complexes.

In [Bj], it is shown that Coxeter complexes and Tits buildings are shellable and Cohen-Macaulay complexes. A generalization of Coxeter complexes for a group with a minimal set of generators is given in [BaR]. Such complexes and their homology give useful representations of the group.

Combinatorial flag varieties were introduced in [BGW] using suitable partial ordering on matroids, and their shellability and other applications are studied in [BGS]. Semibuildings were introduced in [Her]. Other results related to shellability are given in [ER] [Was].

Relations between groups and abstract regular polytopes are explained in [McS]. For generalized polygons, see [Tha1].

8.7 Expanders and Ramanujan graphs

Expanders have been used to construct efficient networks. An important class of expanders consists of the class of Ramanujan graphs. Bruhat-Tits buildings of dimension 1 have played an important role in studying Ramanujan graphs. Higher dimensional Bruhat-Tits buildings have also been used to construct hypergraphs.

For a general introduction to Ramanujan graphs and connections to modular forms, see the book [Li2], surveys [Li3] [Li4]. For constructions of Ramanujan graphs, see also [LPS] [Mors]; for constructions of Ramanujan hypergraphs or

hypercomplexes, see [Li1] [LSV1] [LSV2] [CaSZ] [Sar]. See also [ChL1] [ChL2] for applications of modular forms to Ramanujan graphs.

Construction of Ramanujan graphs is closely related to spectral theory of the graph. Many natural graphs are obtained as quotients of Bruhat-Tits buildings by arithmetic groups. For spectral theory of quotients of Bruhat-Tits buildings of rank 1 (i.e., trees), see [Ef] [Na] [Terr1] [Terr2].

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