# Research Statement

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## 1 Introduction

My research is in geometric group theory and low dimensional topology, and I am primarily interested in understanding, through topological and geometric methods, free groups and their algebraic geometry. The resolution of Tarski's conjecture by Sela and, independently, Kharlampovich-Myasnikov [Sel06, KM06], and Sela's classification of groups elementarily equivalent to a given torsion free hyperbolic group [Sel09b] have made clear the importance of limit groups in geometric group theory and logic. Limit groups[1] have long been central in the study of systems of equations defined over free groups, and their connections to actions of groups on $\mathbb{R}$–trees, Rips' machine, and the JSJ decomposition, underscore their importance.

## 2 Limit groups and Krull dimension

We use $\mathbb{F}$ to stand for a nonabelian free group. *Fully residually free*, or *limit* groups, are the finitely generated groups $L$ with the property that for every finite subset $S$ of $L$ there is a homomorphism $f_S\colon L \to \mathbb{F}$ whose restriction to $S$ is injective.

The sets $\mathrm{Hom}(G, \mathbb{F})$ are analogues to solution sets of collections of polynomials. If $G = \langle x_1, \ldots, x_n \mid r_i(x_j)\rangle$ then $\mathrm{Hom}(G, \mathbb{F})$ is

$$\{(x_1, \ldots, x_n) \in \mathbb{F}^n \mid r_j(x_i) = 1\}$$

The words $r_i$ in the variables $x_i$ are "polynomials" on the affine space $\mathbb{F}^n$, and we take as a basis of closed sets for the Zariski topology on $\mathbb{F}^n$ the zero-sets of words in the variables $x_i$. As in algebraic geometry, the zero-set of a collection of words is the union of finitely many *Zariski* irreducible components, where by irreducible we mean not the union of finitely many closed subsets. Translated into the language of groups, a variety is irreducible if and only if it is $\mathrm{Hom}(L, \mathbb{F})$, for some limit group $L$.

The *elementary* limit groups, which are free, surface, or finite rank free abelian groups, are of special importance. Doubles of free groups along indivisible elements and iterated extensions of centralizers of free groups are basic examples.

[1] Historically called fully residually free groups.

The class of limit groups is closed under taking free products and passing to finitely generated subgroups.

Let $G$ be a finitely generated group, and let $\mathbb{F}$ be a free group. The simplest sets defined over $\mathbb{F}$ are of the form $\mathrm{Hom}(G, \mathbb{F})$. The generators of $G$ play the role of variables and the relations between the generators play the role of polynomials in the variables. We topologize $\mathrm{Hom}(G, \mathbb{F})$ by taking as basic open sets the collection of homomorphisms not vanishing on some fixed $g \in G$. This is simply the Zariski topology. With this topology $\mathrm{Hom}(G, \mathbb{F})$ is irreducible only if $G$ is a limit group.

The Krull dimension of a variety $V$ defined over an algebraically closed field is the length of the longest chain of proper inclusions of irreducible subvarieties of $V$, and we define the Krull dimension of a variety defined over a free group in exactly the same way. Sela conjectured that varieties defined over free groups are finite dimensional.

**Theorem 1** ([Lou09, Lou08]). *Varieties defined over free groups have finite Krull dimension.*

Sela's theory of limit groups generalizes to torsion free hyperbolic groups. The question is also interesting for them.

**Question 1** (Sela). *Are varieties defined over hyperbolic groups finite dimensional?*

Limit groups and irreducible varieties are dual to one another, as varieties and their coordinate rings are dual. A surjection $\pi \colon G \twoheadrightarrow H$ induces an inclusion $\pi^* \colon \mathrm{Hom}(H, \mathbb{F}) \hookrightarrow \mathrm{Hom}(G, \mathbb{F})$, and a sequence of epimorphisms is then dual to a chain of inclusions of varieties. Conversely, any closed subset of $\mathrm{Hom}(G, \mathbb{F})$ is the union of finitely many irreducible varieties associated to quotients of $G$. A sequence of proper epimorphisms of limit groups is a *chain* since it is dual to a chain of inclusions. Theorem 1 is dual to

**Theorem 1′.** There is a function $D(n)$ such that any chain of limit groups starting with $\mathbb{F}_n$ has length at most $D(n)$.

The dimension theorem divides into three steps.

**Step 1:** Strict resolutions are special sequences of epimorphisms which "demonstrate" that a group is a limit group. There is a uniform bound on the length of a strict resolution of a limit group in terms of its rank

**Step 2:** Aligning JSJ decompositions. By step 1 it suffices to consider sequences of limit groups which respect the JSJ decomposition. We deduce the existence of a finite tree of sequences of limit groups, furthermore, the bottom leaves are sequences of elementary limit groups. Assembling this data allows us to lift a dimension bound for "lower" sequences to "higher" sequences.

**Step 3:** Adjoining Roots. This is a technical step in the induction. The tree of sequences derived in step two is actually a tree of nested families of sequences. Each sequence is obtained from a sequence in the same family by "adjoining roots," and a careful analysis of this process is necessary to conclude a dimension bound for sequences at higher levels from a dimension bound for sequences as lower levels.

## 2.1 Strict resolutions

A morphism of nonabelian limit groups $\pi\colon L \to L'$ is *strict* if it has nonabelian image and, for each finite subset $S$ of each freely indecomposable free factor $K$ of $L$, there is an automorphism $\varphi \in \mathrm{Aut}(K)$ such that $\pi \circ \varphi$ restricted to $S$ is injective. One should think of $L$ as being fully residually $L'$, but rather than allowing arbitrary homomorphisms $L \to L'$, we only allow maps of the form $\pi \circ \varphi$. Strictness may also be detected by checking if the morphism embeds certain subgroups defined geometrically in terms of the JSJ decomposition (see Subsection 3). The simplest nontrivial example is the canonical retraction of a double of a free group along an indivisible element to one of the factors. Any given finite subset embeds if the retraction is preceded by a sufficiently high power of the Dehn twist in the edge group. A *resolution* of a limit group $L$ is a sequence of epimorphisms

$$L \twoheadrightarrow L_1 \twoheadrightarrow \ldots \twoheadrightarrow L_k$$

of limit groups. A resolution is *strict* if all its morphisms are strict.

**Theorem 2** ([Lou09, Hou08])**.** *Any strict resolution of a limit group $L$ has length at most $3\,\mathrm{rk}(L)$.*

The proof of Theorem 2 relies on linearity of free groups. Strictness implies the existence of a chain of proper inclusions of irreducible components of (suitable) representation varieties, and the bound follows from finite dimensionality of ordinary varieties. Hyperbolic groups are not in general linear, and a proof along these lines for resolutions by Γ–limit groups cannot exist. Limit groups may be thought of as geometric limits of free groups [CG05], and limit groups are precisely the accumulation points of the set of marked free groups in the space of marked groups, in the Gromov-Hausdorff/Chabauty topology. Houcine proves that the set of limit groups in this space has finite Cantor-Bendixon rank. In this interpretation a strict map $L \twoheadrightarrow L'$ shows that $L$ is a limit of marked groups isomorphic to $L'$.

**Question 2.** *Is there a proof of Theorem 2 that doesn't use representation varieties?*

Limit groups over hyperbolic groups are defined in the same way as limit groups over free groups.

**Question 3.** *Is Theorem 2, with the linear bound replaced by some other function of the rank, true for resolutions by Γ–limit groups?*

A torsion free hyperbolic group Γ has *uniformly bounded strict resolutions* if the lengths of strict resolutions of $\mathbb{F}_n$ by Γ–limit groups have uniformly bounded lengths, depending on $n$.

## 2.2 Aligning JSJ decompositions

The JSJ decomposition of a three-manifold is concerned with *essential* (two-sided, embedded, incompressible) tori. There is a canonical collection of disjoint essential tori in $M$ such that the complementary components are either Seifert-fibered or atoroidal. Every essential torus in a three-manifold $M$ corresponds either to one of the canonical tori or to a torus in a Seifert-fibered component. Group theoretically, an essential torus corresponds to a splitting of the fundamental group over $\mathbb{Z}^2$, and all such splittings of $M$ can be read from the JSJ decomposition [JS79, Joh79].

An abelian JSJ decomposition [DS99, FP06, RS97] of a one-ended limit group $G$ is, loosely speaking, a graph of groups decomposition encoding the entire collection of splittings of $G$ over abelian edge groups.

The key step in Theorem 1 is to reduce the analysis of arbitrary chains to an analysis of chains of limit groups whose maps are natural with respect to Grushko and JSJ decompositions. An epimorphism of free products is easiest to analyze if it maps freely indecomposable free factors to freely indecomposable free factors, and sequences of epimorphisms that break into free products of sequences are simpler than arbitrary sequences.

A similar philosophy holds for JSJ decompositions. Let $G$ and $H$ be one-ended limit groups. An epimorphism $\varphi\colon G \to H$ *respects* JSJ decompositions if it maps edge groups to edge groups, vertex groups to vertex groups, is a one-to-one correspondence on vertex groups, respects types (rigid vertex groups are mapped to rigid vertex groups, etc.) and maps each QH vertex group of $G$ isomorphically to the corresponding QH vertex group of $H$. The following is a simplified version of the alignment property needed for Theorem 1 [Lou09].

**Theorem 3** (Alignment theorem, simplified version)**.** *Consider $\mathcal{S}_n = \{\mathbb{F}_n \twoheadrightarrow L_1 \twoheadrightarrow \cdots \twoheadrightarrow L_k\}$, the set of all chains of limit groups with rank bounded by $n$. Let $\mathcal{A}_n \subset \mathcal{S}_n$ be the subset such that each map respects both Grushko decompositions and the JSJ decompositions of freely indecomposable free factors. Then chains in $\mathcal{S}_n$ are of uniformly bounded length if and only if chains in $\mathcal{A}_n$ are of uniformly bounded length.*

After using this theorem to reduce the analysis of general sequences to an analysis of the small subset of sequences which behave well with respect to graph of groups decompositions, we restrict our attention to sequences of vertex groups of JSJ decompositions. Sequences of vertex groups are, after handling a number of technicalities, of lower complexity than the original sequences. Uniformly bounded strict resolutions are a key ingredient in the alignment theorem.

**Question 4.** *Does the alignment theorem hold for limit groups over a torsion free hyperbolic group having uniformly bounded strict resolutions?*

Two main problems are left if the alignment theorem holds for a group $\Gamma$. First, there is the question of termination. Iterating this procedure, we would like to discover a collection of sequences having demonstrably bounded length. For a free group, the process terminates because the analysis lattices are uniformly finite, and the chains at at the end of the process consist of elementary limit groups.

For a hyperbolic group $\Gamma$ with uniformly bounded strict resolutions, repeated application of the alignment theorem yields sequences of epimorphisms of subgroups of $\Gamma$. Subgroups of hyperbolic groups have little structure and new methods are necessary in order to control the lengths of such sequences. See Section 4.

The second problem is that the sequences of vertex groups are not necessarily epimorphisms. Each group in a derived sequences is obtained from the corresponding group earlier in the sequence by *adjoining roots.* This makes it hard to lift a dimension bound for a collection of sequences of vertex groups to a dimension bound for the sequence itself.

### 2.3 Adjoining roots

Let $\mathcal{C}$ be some class of finitely generated groups. Given $g \in G \in \mathcal{C}$, is there an overgroup $H \in \mathcal{C}$ of $G$ such that $g$ has an $n$–th root in $H$? For the inductive step in Theorem 1 we need to understand how adjoining roots works when $\mathcal{C}$ is the class of limit groups. It is possible to say something completely general about finitely generated $G$ and $H$ as long as $H$ has no $\mathbb{Z}_2$ torsion.

Let the *Scott complexity* of a finitely generated group be the lexicographically ordered pair $sc(G) = (p + q, q)$, where $p + q$ is the maximal number of factors in a free product decomposition of $G$ and $q$ is the largest rank of a free free factor of $G$.

**Theorem 4** ([Lou08]). *Suppose an inclusion $G \hookrightarrow H$ of finitely generated torsion free groups extends to an epimorphism*

$$G' = \langle G, h \mid h^k = g\rangle \twoheadrightarrow H$$

*for some indivisible $g \in G$. Then at least one of the following holds:*

1. *$\langle g\rangle$ is a free factor of $G$*
2. *$g$ is conjugate into a freely indecomposable free factor of $G$*
3. *$sc(G) > sc(H)$.*

Baumslag [Bau65] proves this for free $G$ and $H$, but Theorem 4 does not follow from this special case. If neither of the last two cases holds then there is a nice graph of spaces representing the map $G' \to H$. A careful analysis of this space then yields the first case of the theorem. If the second case holds then in some special cases, such as the generalization to JSJ decompositions of limit groups, we can still learn something about the adjunction [Lou09].

There is a complexity, depending only on the rank, such that if $G$ and $H$ are limit groups, and $G$, $H$, and $G'$ are related as above, and furthermore $G' \twoheadrightarrow H$ factors through a limit group $K$, then either $H$ is less complicated than $G$ or $G$ and $H$ have the "same" analysis lattices. We then relate the analysis lattices of

$H$ and $G$ to the lattice of $K$, and show that the groups in the analysis lattice of $K$ are obtained, roughly, from the corresponding groups in the analysis lattice of $H$ by adjoining roots. If the complexities are equal then the map $K \to H$ must be an isomorphism.

# 3 Accessibility and analysis lattices

Let $G$ be a finitely generated freely indecomposable group. A *hierarchy* for $G$ is a rooted tree of groups $\mathcal{H}$ defined as follows: $G$ is the root of $\mathcal{H}$. If $K \in \mathcal{H}$ is defined let the children of $K$ be the freely indecomposable free factors of a graphs of groups decomposition of $K$ over finitely generated abelian edge groups, subject to the requirement that if $K$ is the fundamental group of a closed surface or is free abelian then $K$ has no children.

The *analysis lattice* of a limit group is the hierarchy associated to passing to the abelian JSJ decomposition at every step.

An *elementary family* in a group $G$ is a collection of subgroups, closed under passing to subgroups and conjugation, with the property that any two elements either intersect in a finite subgroup or the subgroup they generate fixes a point or an end whenever acting on a tree

**Theorem 5** (Delzant-Potyagailo [DP01])**.** *Let $G$ be finitely presented, $\mathcal{C}$ an elementary family of subgroups of $G$. Then there is a finite hierarchy of $G$ corresponding to splittings over $\mathcal{C}$.*

Unfortunately the proof of Theorem 5 in [DP01] contains a fundamental error (Private communication.). Nicholas Touikan and I are currently writing up the following weaker theorem.

**Theorem 6** (Accessibility for torsion free relatively hyperbolic groups)**.** *Let $G$ be a torsion free relatively hyperbolic group, and consider a hierarchy associated to passing to vertex groups of the abelian* JSJ *decomposition relative to the family of parabolic subgroups of $G$. Then $\mathcal{H}$ is finite.*

*If $G$ is toral relatively hyperbolic (hyperbolic relative to a family of finitely generated free abelian subgroups) then there is a number $H(G)$ such that any hierarchy of $G$ has at most $H(G)$ elements.*

Touikan and I use a variation on the technique of [DP01] to produce a sequence of orbifold groups of finitely many types, each of which corresponds to an element of a hierarchy. Finiteness of combinatorial types enables the construction finitely many "universal" groups $U_1, \ldots, U_n$, and each element of a hierarchy is a specialization of one of the $U_i$ given by Dehn filling free abelian subgroups. The Delzant/Dahmani/Sela theorem bounding the number of images of each $U_i$ in $G$ up to a certain natural equivalence ("bending", or in Dahmani's terminology, "accidental parabolics") allows us to argue that the hierarchy must have been finite.

**Question 5.** *Is Theorem 6 true for subgroups of (relatively) hyperbolic groups?*

Dahmani's theorem that a fixed group admits only finitely many images in a relatively hyperbolic group without accidental parabolics does not use limiting arguments, and since the presentations of subgroups obtained through Delzant/Potyagailo are of such a specific type it may be possible to extract a proof that doesn't use the full strength of Dahmani's theorem. The difficulty in adapting our technique is that subgroups of hyperbolic groups do not have to be co-Hopfian or finitely presented.

Say that a group $G$ is *accessible* if it admits a graph of groups decomposition with finite edge groups and one-ended vertex groups. Dunwoody has an example of a finitely generated, non-finitely-presentable, inaccessible group with unbounded torsion [Dun93]. Unbounded torsion is the only obstruction to accessibility [Lin83]. Call a group *strongly accessible* if every hierarchy of splittings over an elementary family is finite. The only examples of non-strongly accessible groups that I know of also have unbounded torsion. The following is new, but unsurprising.

**Example 7** ((in preparation))**.** *There is an infinite sequence $G = G_0 > G_1 > G_2 > \cdots$ of finitely generated groups, each with unbounded torsion, such that $G_{i+1}$ is a one-ended vertex group in a graph of groups decomposition of $G_i$ over two-ended edge groups.*

**Question 6.** *Is every finitely generated group with bounded torsion strongly accessible?*

## 4 Subgroups/non-quasiconvexity

Another difficulty in generalizing the dimension theorem is that the base cases of the induction are complicated. The simplest varieties defined over a free group correspond to closed surfaces, free abelian groups, and free groups. Over a general hyperbolic group there are many more possibilities for terminal leaves of the analysis lattice. These must be analyzed as well.

**Question 7.** *Fix $\Gamma$ torsion free hyperbolic. Consider sequences of proper epimorphisms*

$$\mathbb{F}_n \twoheadrightarrow H_1 \twoheadrightarrow \cdots \twoheadrightarrow H_k,$$

*$H_i < \Gamma$. Is $k$ bounded by a function of $n$?*

It follows from a weak form of the alignment theorem used in the proof of Krull dimension for limit groups that we only have to consider sequences of one-ended subgroups such that no map above factors through a free product. In the locally-quasiconvex case the question is resolved by appealing to the theorem of Kapovich and Weidmann that if $\Gamma$ is locally quasiconvex then there are only finitely many one-ended subgroups of rank $n$ [KW04].

The real difficulty lies in understanding non-quasiconvex subgroups. There is at least one class of groups for which non-quasiconvex subgroups are well understood, and that is the fundamental groups of closed hyperbolic three manifolds. In this case, the covering theorem and tameness [Thu02, Can96, Ago04, CG06] imply that any non-quasiconvex subgroup is a closed surface group that the manifold virtually fibers over.

Using the carrier graph method from [KW04] and the characterization of nonquasiconvex subgroups of hyperbolic three manifold groups, Biringer, Souto, and I prove the following.

**Theorem 8.** *Let $\Gamma$ be the fundamental group of a closed orientable hyperbolic three-manifold $M$. Fix $n$. Then there are subgroups $H_1, \ldots, H_k$ such that any one-ended subgroup $K$ generated by at most $n$ elements in $\Gamma$ is either conjugate to one of the $H_i$ or the cover of $M$ associated to $K$ fibers over the circle (or an interval) with fiber a surface of genus bounded by a function of $n$.*

This theorem organizes the collection of subgroups of the fundamental group of a closed orientable hyperbolic three manifold group into finitely many families, up to finitely many exceptional cases. In some sense it is a "uniform" generalization, for hyperbolic three manifold groups, of the theorem of Gromov/Delzant/Sela/Thurson/Dahmani that there are only finitely many conjugacy classes of subgroups isomorphic to a given one-ended group [Gro87, RS94, Sel97, Thu02].

Theorem 8 divides the family of one-ended subgroups of a given rank into naturally occurring families, and should allow us to resolve Question 7 for hyperbolic three-manifold groups.

# 5 Indefinability of the set of simple closed curves

Let $\mathbb{F}_2 = \langle a, b \rangle$ be the nonabelian free group of rank two. An element of $\mathbb{F}$ is primitive if and only if its commutator with some other element is conjugate to $[a, b]$, that is, the sentence

$$\varphi(x) = \exists y, z \mid [x, y]^z = [a, b]^{\pm 1}$$

is satisfied by $x$ if and only if $x$ is primitive. For higher rank, it is an unpublished theorem of Bestvina and Feighn that there is no sentence which is true only on primitive elements.

**Question 8.** *Is there a first order sentence $\varphi(x, \pi_1(\Sigma))$ in the language of groups, with constant symbols from $\pi_1(\Sigma)$, such that $\varphi(x, \pi_1(\Sigma))$ is true only if $x$ is a representable by a simple closed curve?*

Chloe Perin and I are attempting to use techniques from [Sel09a] to answer this question in the negative. The set of primitive elements in $\mathbb{F}_2$ is a *Diophantine* set, that is, a set defined by a formula with only one existential quantifier. In general definable sets require more quantifier to specify, but Sela has proven that given a general definable set in a free or surface group there is an *envelope*, a finite family of Diophantine sets, containing all "generic" points. The construction appears to imply that any definable set containing only simple closed curves is contained in a finite union of sets of the form

$$Mod(\Sigma_1) \circ \cdots \circ \mathrm{Mod}(\Sigma_n)([c])$$

where $[c]$ is a conjugacy class of simple closed curves and $\mathrm{Mod}(\Sigma_i)$ is the mapping class group of a *proper* subsurface of $\Sigma$. By Bestvina-Fujiwara [BF07] there is an unbounded quasihomomorphism $\mathrm{Mod}(\Sigma) \to \mathbb{R}$ which is bounded on subgroups fixing subsurfaces, and the definable set therefore can't exhaust the set of simple closed curves.

# 6 The Magnus property

Limit groups are used to study the *first order* properties of free and hyperbolic groups. Loosely speaking, the first order theory of a group is the set of all sentences, using only the language of groups and ordinary symbols from logic, that are true of the group. Two groups are said to be *elementarily equivalent* if they have the same first order theories, and one important task is the classification of all groups elementarily equivalent to a given group.

Let $G$ be a limit group and suppose $H < G$ is a retract of $H$. Then $G$ is said to be a tower over $H$ if the retraction is strict and the $G$ has a decomposition resembling the JSJ in which $H$ is elliptic, and the vertex groups not containing $H$ are all surfaces with boundary or are abelian, and are all attached to $H$ by abelian edge groups. A limit group $G$ is a tower if it is the top level of a sequence

$$G_0 < G_1 < \cdots < G_n = G$$

such that $G_0$ is a free product of elementary limit groups and for each $i$ $G_i$ is a tower over $G_{i-1}$. Geometrically, at stage $i$ we glue surfaces with boundary and tori to $G_i$ to obtain $G_{i+1}$, in such a way that there is a retraction $G_{i+1} \to G_i$ sending surfaces to nonabelian subgroups. A *hyperbolic tower* is a tower where all stories of the tower are made of surfaces. Hyperbolic towers are elementarily equivalent to free groups are [Sel06].

Let $G$ be a group and $S \subset G$. We denote the normal closure of $S$ by $\langle\langle S \rangle\rangle$. A group $G$ has the *Magnus property* if, for all $a, b \in G$, if $\langle\langle a \rangle\rangle = \langle\langle b \rangle\rangle$ then either $a \sim b$ ($a$ and $b$ are conjugate) or $a \sim b^{-1}$. We call a pair of elements having the same normal closure a *Magnus pair*. The Magnus property (or lack thereof) is shared by elementarily equivalent groups, and since free groups have the Magnus property, all hyperbolic towers do too.

Nicholas Touikan and I can show the following.

**Theorem 9** (Louder-Touikan)**.** *If $G$ is a tower then $G$ has the Magnus property. If $a$ and $b$ have conjugate images under every map $G \to \mathbb{F}$, then $a$ and $b$ are conjugate in $G$.*

*Let $G$ be a tower, $g \in G$ an indivisible element. If $g$ is not contained in a noncyclic abelian subgroup, then there is a sequence $f_n \colon G \to \mathbb{F}$ converging to $G$ such that $f_n(g)$ is indivisible.*

This is a generalization of [Lio03], which shows that iterated extensions of centralizers in a free group have the Magnus property. The converse is false: There exists a freely indecomposable limit group $G$, which is *not* a tower, and a sequence $f_n\colon G \to \mathbb{F}$ converging to $G$, such that if $a$ and $b$ are nonconjugate then $f_n(a)$ and $f_n(b)$ are eventually nonconjugate, likewise for indivisibility of images of indivisible elements.

The Magnus property can be reinterpreted in topological terms. Let $S_1$ and $S_2$ be two spheres with boundary, and let $s_1$ and $s_2$ be two circles. Glue all but one boundary component of $S_1$ to $s_1$ in some way, and the remaining to $s_2$, and glue all but one boundary component of $S_2$ to $s_2$ in some way, and the remaining one to $s_1$. Call the resulting complex $X$. For a free group, the Magnus property is simply the assertion that the restrictions of a continuous map $X \to Y$, $Y$ a graph, to $s_1$ and $s_2$ are freely homotopic, up to orientation. It may not be the case that $\pi_1(X)$ is a limit group, but this can be arranged by choosing $S_i$ sufficiently complicated and carefully choosing orientations when gluing.

**Problem 1.** *Describe* $\mathrm{Hom}(\pi_1(X), \Gamma)$*, where* $\Gamma$ *is a limit group, a torsion free hyperbolic group, or a free product.*

Let $S$ be a closed orientable surface bounding a handlebody $B$, and let $i$ be the inclusion map. Stallings showed that every map $\pi_1(S) \to \mathbb{F}$ factors, up to conjugacy, as $f \circ i_* \circ \varphi$, where $f\colon \pi_1(B) \to \mathbb{F}$ is a map of free groups and $\varphi$ is an element of the mapping class group of $S$. Let $X$ be a space obtained by identifying boundary components of a compact genus 0 orientable surface $S$. In a free group, Magnus's theorem and the following can be used to give a Stallings-like characterization of maps from $\pi_1(X)$ to $\mathbb{F}$.

**Theorem 10** (Lyndon's Identity Theorem [Lyn72])**.** *Let* $f\colon \pi_1(X) \to \mathbb{F}$*. Then* $f$ *kills a simple closed curve in* $S \subset X$*.*

The identity theorem holds for surface groups [How04], but not for all groups elementarily equivalent to a free group. For instance, by carefully choosing orientations, the fundamental group of a space obtained by gluing identifying a maximal collection of nonparallel nonseparating curves in an orientable surface is a hyperbolic tower.

**Question 9.** *For which limit groups is the identity theorem true? Are there any* proper *limit quotients of* $\pi_1(X)$ *that don't kill any simple closed curves?*

If not then it is likely that a limit group has only finitely many exceptional Magnus pairs.

Lyndon's proof of the identity theorem is a variation on the proof of the Freiheitssatz, and the extension to surface groups requires a reformulation of the Freiheitssatz for surface groups [HS09]. The following also seems like an interesting formulation of the Freihetssatz for surface groups.

**Question 10.** *Let* $S$ *be a compact closed surface, and let* $c$ *be a (geodesic) curve. Let* $S_c$ *be the subsurface of* $S$ *obtained by cutting the smallest subsurface of* $S$ *containing* $c$ *along all arcs in* $S_c$ *intersecting* $c$ *once. Does the smallest subsurface containing any element of the normal closure of* $c$ *also contain* $S_c$*?*

**Question 11** (Sela). *Does a torsion free hyperbolic group have the Magnus property, up to finitely many exceptional pairs?*

I construct counterexamples to this question in the following geometric way. Hyperbolic two-bridge knot groups, by [SWW09], have infinitely many distinct conjugacy classes of elements that normally generate and whose geodesic representatives don't extend arbitrarily far into the cusp. By carefully Dehn filling a hyperbolic two-bridge knot complement we can guarantee that the images of these elements are nonconjugate, and we obtain infinitely many hyperbolic three-manifold groups, each with an infinite collection of nonconjugate elements, each of which normally generates the fundamental group.

**Question 12.** *Do Magnus pairs come in natural families, i.e., in a fixed finitely generated hyperbolic group, is there a uniform upper bound to the minimal number of conjugates of $a$ which have to be multiplied together to get $b$ if $a$ and $b$ form a Magnus pair?*

# 7 Uniform conjugators

Fix a finite generating set $S$ for a group $G$. We say that two homomorphisms $f, g\colon G \to \Gamma$ are $n$–related if $f(x)$ and $g(x)$ are conjugate for all $x$ in the ball of radius $n$ about the identity of the Cayley graph of $G$ with respect to $S$.

**Theorem 11** ([BV09]). *Let $\Gamma$ be a torsion free hyperbolic group. For all maps $f\colon \mathbb{F}_k \to \Gamma$ there exists $n(f)$ such that if $f$ and $g$ are $n(f)$–related then $g$ is conjugate to $f$.*

**Question 13.** *Fix $k$. Is* $\sup\{n(f) \mid f\colon \mathbb{F}_k \to \Gamma\} < \infty$*?*

Take two copies $H$ and $K$ of $\mathbb{F}_k$ and an isomorphism $\varphi\colon H \to K$. The sets

$$X_n = \{(f, g) \in \mathrm{Hom}(H, \Gamma) \times \mathrm{Hom}(K, \Gamma) \mid f \text{ is } n\text{–related to } g \circ \varphi\}$$

form a nested family $X_1 \supset X_2 \supset X_3 \cdots$ of *existential sets*. Theorem 1 is a strong kind of *descending chain condition* for varieties. In general, descending chains of existential sets do not stabilize, and a positive answer to Question 13 is equivalent to the sequence $X_n$ stabilizing. A good test case for Question 13 is that of a free group.

# 8 Nielsen Equivalence in surface groups

**Theorem 12** ([Lou10]). *Any generating set for a closed surface group is either reducible (i.e., a sequence of Nielsen moves can be applied to eliminate a generator) or equivalent to a standard generating set.*

Zieschang proves that minimal generating sets are all equivalent, as long as the surface is orientable and doesn't have genus 3 [Zie70]. Theorem 12 can also be thought of as the Wiegold conjecture for surface groups.

*Books of I–bundles*, spaces obtained by gluing thickened surfaces to solid tori, are perhaps the simplest three manifolds. An attempt to prove Theorem 12 for books of $I$–bundles led to generating sets that appear to be irreducible but not of minimal cardinality.

**Problem 2.** *Show that there are non-minimal irreducible generating sets for books of I–bundles.*

One has to find a way to prove Nielsen inequivalence in this context, which is certainly more tractable than in general (where it is not decidable).

# 9 Simple loop conjecture for limit groups and representations

One important question in the topology of three-manifolds is the simple loop conjecture, which states that the map on fundamental groups induced by a two-sided immersed closed surface in a three manifold is either injective or it kills a simple closed curve. Minsky asked whether or not the simple loop conjecture holds for representations of surface groups, and I can produce counterexamples, for genus at least four, in the category of limit groups, and, with the right generalization of strictness to $SL(2, \mathbb{C})$, examples in all genera.

**Theorem 13** ([Lou])**.** *Let $S$ be the fundamental group of a closed orientable surface of genus $g > 1$, and let $k$ be a non-negative integer. Then $S$ admits a noninjective map $\rho\colon S \to SL(2, \mathbb{C})$ which doesn't kill any closed curves of self-intersection number at most $k$. Furthermore, $\rho$ may be chosen so that for all $h \in S$ $\rho(h)$ is either trivial or hyperbolic.*

Manning and Cooper have constructed examples containing parabolics. To construct limit groups that don't satisfy the simple loop conjecture we take certain one-relator limit quotients of surface groups, and by Hempel's theorem no simple closed curves die [Hem90]. This construction works in genus at least four, since limit groups embed in $SL(2, \mathbb{C})$ [BG09]. For genus at most three these groups are not limit groups, there is no version of Hempel's theorem, and we have to be more careful with the construction.

For higher intersection numbers Hempel's theorem doesn't hold and we use a limiting procedure and an argument using low complexity limit groups to avoid complicated cancellation arguments.

# 10 Graphs of free groups

One outgrowth of [Lou08] is a nice model for subgroups of free groups and their intersections. Ben McReynolds and I construct a simple model for the amalgam $H *_{H \cap K} K$, where $H$ and $K$ are finitely generated subgroups of a free group. The construction is formally similar but more transparent than the one used by Dicks to show equivalence of the Hanna Neumann conjecture and the Amalgamated

Graph Conjecture [Dic94]. Kent used similar ideas to prove a strong form of Burns' inequality for the Hanna Neumann conjecture. We prove the following, improving upon [Bur71].

**Theorem 14** ([LM09, Ken09])**.** *If two rank two subgroups of a free group generate a subgroup of rank three then their intersection is cyclic (possibly trivial).*

Culler and Shalen use this result in volume estimates for four-free (i.e., every subgroup generated by at most four elements is free) hyperbolic three-manifolds [CS09]. Another consequence of the method is a very transparent proof of Burns' theorem that the intersection of two rank two subgroups of a free group has rank at most two.

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