Logic rules.

(2) To prove a statement $H \Rightarrow C$, assume the negation of C (RAA hypothesis) and deduce an absurd statement, using the hypothesis H if needed in your deduction.
(3) The statement "$\neg(\neg S)$" means the same as "S".
(4) The statement "$\neg[H \Rightarrow C]$" means the same thing as "$H \wedge \neg C$".
(5) The statement "$\neg[S_1 \wedge S_2]$" means the same as "$[\neg S_1 \vee \neg S_2]$".
(6) The statement "$\neg[\forall x S(x)]$" means the same as "$\exists x \neg S(x)$".
(7) The statement "$\neg[\exists x S(x)]$" means the same as "$\forall x \neg S(x)$".
(8) If $P \Rightarrow Q$ and P are steps in a proof, then Q is a justifiable step.
(9) (a) $[[P \Rightarrow Q] \wedge [Q \Rightarrow R]] \Rightarrow [P \Rightarrow R]$.
  (b) $[P \wedge Q] \Rightarrow P$, $[P \wedge Q] \Rightarrow Q$.
  (c) $[\neg Q \Rightarrow \neg P] \Leftrightarrow [P \Rightarrow Q]$.
(10) For every statement P, "$P \vee \neg P$" is a valid step in a proof (law of excluded middle).
(11) Suppose the disjunction $S_1 \vee \ldots \vee S_n$ is already a valid step in a proof. Suppose that proofs of C are carried out from each of the *case assumptions* $S_1, \ldots, S_n$. Then C can be concluded as a valid step in the proof (proof by cases).
(12) (a) $\forall X\ (X = X)$.
  (b) $\forall X\ \forall Y\ (X = Y \Leftrightarrow Y = X)$.
  (c) $\forall X\ \forall Y\ \forall Z((X = Y \wedge Y = Z) \Rightarrow X = Z)$.
  (d) If $X = Y$ and $S(X)$ is a statement about X, then $S(X) \Leftrightarrow S(Y)$.

Incidence axioms.

(1) For every point P and for every point Q not equal to P, there exists a unique line ℓ incident with P and Q. We write $\ell = \overleftrightarrow{PQ}$.
(2) For every line ℓ, there exist at least 2 distinct points incident with ℓ.
(3) There exist 3 distinct points with the property that no line is incident with all 3 of them.

Parallelism properties.

- *Elliptic parallel property.* If ℓ and m are 2 lines, then there is some point P that is incident with ℓ and m.
- *Euclidean parallel property.* For every line ℓ and every point P not incident with ℓ, there exists a unique line that is incident with P and parallel to ℓ.
- *Hyperbolic parallel property.* For every line ℓ and every point P not incident with ℓ, there exist at least 2 lines that are incident with P and parallel to ℓ.

A model of incidence geometry satisfying the Euclidean parallel property is called a *affine plane*. A model of incidence geometry satisfying the hyperbolic parallel property is called a *hyperbolic plane*. A model of incidence geometry satisfying the elliptic parallel property and the following axiom:

- *Strengthened axiom I2.* For every line ℓ, there exist at least 3 distinct points incident with ℓ.

is called a *projective plane*.

Betweenness axioms

(1) If $A * B * C$, then A, B, and C are 3 distinct points all lying on the same line, and $C * B * A$.

(2) Given any 2 distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A * B * D$, $B * C * D$, and $B * D * E$.
(3) If A, B, and C are 3 distinct points lying on the same line, then 1 and only 1 of the points is between the other 2.
(4) For every line ℓ and for any 3 points A, B, and C not lying on ℓ, (i) if A and B are on the same side of ℓ and if B and C are on the same side of ℓ, then A and C are on the same side of ℓ; (ii) if A and B are on opposite sides of ℓ and if B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ; and (Corollary, iii) if A and B are on opposite sides of ℓ and if B and C are on the same side of ℓ, then A and C are on opposite sides of ℓ.

Some propositions.

2.1. If ℓ and m are distinct lines that are not parallel, then ℓ and m have a unique point in common.
2.2. There exist 3 distinct lines that are not concurrent.
2.3. For every line, there is at least 1 point not lying on it. (Alternately, "No line is incident with all points.")
2.4. For every point, there is at least 1 line not passing through it. (Alternately, "No point is incident with all lines.")
2.5. For every point P, there exist at least 2 distinct lines that are incident with P.
3.1. For any two points A and B, (i) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$, and (ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.
3.2. Every line bounds exactly two half-planes, and these half-planes have no point in common.
3.3. Given $A * B * C$ and $A * C * D$. Then $B * C * D$ and $A * B * D$. **Corollary**: Given $A * B * C$ and $B * C * D$. Then $A * B * D$ and $A * C * D$.

Some definitions

Segment: If A and B are 2 distinct points, and P is any point, then P is incident with segment $\overline{AB}$ if and only if $P = A$ or $A * P * B$ or $P = B$.

Ray: If A and B are 2 distinct points, and P is any point, then P is incident with ray $\overrightarrow{AB}$ if and only if P is incident with segment $\overline{AB}$ or $A * B * P$.

Sides: If ℓ is a line, and A and B are 2 points that are not incident with ℓ, then A and B are *on the same side* of ℓ if and only if $A = B$ or there is no point that is incident with segment $\overline{AB}$ and ℓ; and A and B are *on opposite sides* of ℓ if and only if $A \neq B$ and there is a point that is incident with segment $\overline{AB}$ and ℓ.

Half-planes: If ℓ is a line and A is a point that is not incident with ℓ, then the set of all points that are on the same side of ℓ as A is called a *half-plane* bounded by ℓ.