Congruence axioms

(1) If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $\overline{AB} \cong \overline{A'B'}$.
(2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Moreover, every segment is congruent to itself.
(3) Segment addition: If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.
(4) Given any $\angle BAC$ (where, by the definition of "angle", $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$) and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\angle B'A'C' \cong \angle BAC$.
(5) If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.
(6) SAS test for congruence: If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

The following versions of axioms C2 and C5 may be more convenient in practice.

(2′) Congruence of segments satisfies the following properties. Let s, s', and s'' be any 3 segments.
  (a) Reflexivity: $s \cong s$.
  (b) Symmetry: If $s \cong s'$, then $s' \cong s$.
  (c) Transitivity: If $s \cong s'$ and $s' \cong s''$, then $s \cong s''$.
(5′) Congruence of segments satisfies the following properties. Let α, β, and γ be any 3 angles.
  (a) Reflexivity: $\alpha \cong \alpha$.
  (b) Symmetry: If $\alpha \cong \beta$, then $\beta \cong \alpha$.
  (c) Transitivity: If $\alpha \cong \beta$ and $\beta \cong \gamma$, then $\alpha \cong \gamma$.

Definitions

Interior (angles): Given an angle $\angle CAB$, define a point D to be in the *interior* of $\angle CAB$ if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C.
Betweenness (rays): Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\angle CAB$.
Interior (triangles): The *interior* of a triangle is the intersection of the interiors of its three angles. A point is *exterior* to the triangle if it is not in the interior and does not lie on any side of the triangle.
Congruence (triangles): Triangles $\triangle ABC$ and $\triangle DEF$ are *congruent* if there exists a one-to-one correspondence between their vertices such that corresponding sides are congruent and corresponding angles are congruent. We will use the notation $\triangle ABC \cong \triangle DEF$ to indicate not only that these triangles are congruent but that a correspondence demonstrating that congruence is such that A corresponds to D, B to E, and C to F (i.e., the order in which we write the vertices matters).
Order (segments): $\overline{AB} < \overline{CD}$ (or $\overline{CD} > \overline{AB}$) means that there exists a point E between C and D such that $\overline{AB} \cong \overline{CE}$.
Order (angles): $\angle ABC < \angle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle ABC \cong \angle GEF$.
Acute; obtuse: An angle is *acute* if it is less than a right angle, *obtuse* if it is greater than a right angle.

Some results

3.4. If $C * A * B$ and ℓ is the line through A, B, and C, then for every point P lying on ℓ, P lies either on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

- Pasch's theorem: If A, B, C are distinct noncollinear points and ℓ is any line intersecting $\overline{AB}$ in a point between A and B, then ℓ also intersects either $\overline{AC}$ or $\overline{BC}$. If C does not lie on ℓ, then ℓ does not intersect both $\overline{AC}$ and $\overline{BC}$.

3.5. Given $A * B * C$. Then $\overline{AC} = \overline{AB} \cup \overline{BC}$ and B is the only point common to segments $\overline{AB}$ and $\overline{BC}$.
3.6. Given $A * B * C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.
3.7. Given an angle $\angle CAB$ and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of $\angle CAB$ if and only if $B * D * C$.
3.8. If D is in the interior of $\angle CAB$, then (a) so is every other point on ray $\overrightarrow{AD}$ except A; (b) no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\angle CAB$; and (c) if $C * A * E$, then B is in the interior of $\angle DAE$.
- Crossbar theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment $\overline{BC}$.

3.9. (a) If a ray r emanating from an exterior point of $\triangle ABC$ intersects side $\overline{AB}$ in a point between A and B, then r also intersects side $\overline{AC}$ or side $\overline{BC}$. (b) If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if it does not pass through a vertex, it intersects only one side.
- Corollary to SAS: Given $\triangle ABC$ and segment $\overline{DE} \cong \overline{AB}$, there is a unique point F on a given side of line $\overleftrightarrow{DE}$ such that $\triangle ABC \cong \triangle DEF$.

3.10. If in $\triangle ABC$ we have $\overline{AB} \cong \overline{AC}$, then $\angle B \cong \angle C$.
3.11. Segment subtraction: If $A * B * C$, $D * E * F$, $\overline{AB} \cong \overline{DE}$, and $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$.
3.12. Given $\overline{AC} \cong \overline{DF}$, then for any point B between A and C, there is a unique point E between D and F such that $\overline{AB} \cong \overline{DE}$.
3.13. Segment ordering: (a) Exactly one of the following three conditions holds (*trichotomy*): $\overline{AB} < \overline{CD}$, $\overline{AB} \cong \overline{CD}$, or $\overline{AB} > \overline{CD}$. (b) If $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} < \overline{EF}$. (c) If $\overline{AB} > \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} > \overline{EF}$. (d) If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$ (*transitivity*).
3.14. Supplements of congruent angles are congruent.
3.15. (a) Vertical angles are congruent to each other. (b) An angle congruent to a right angle is a right angle.
3.16. For every line ℓ and every point P there exists a line through P perpendicular to ℓ.
3.17. ASA test for congruence: Given $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $\overline{AC} \cong \overline{DF}$. Then $\triangle ABC \cong \triangle DEF$ [1].
3.18. If in $\triangle ABC$ we have $\angle B \cong \angle C$, then $\overline{AB} \cong \overline{AC}$ and $\triangle ABC$ is isosceles.
3.19. Angle addition: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\angle CBG \cong \angle FEH$, and $\angle GBA \cong \angle HED$. Then $\angle ABC \cong \angle DEF$.
3.20. Angle subtraction: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{EH}$ and $\overrightarrow{EF}$, $\angle CBG \cong \angle FEH$, and $\angle ABC \cong \angle DEF$. Then $\angle GBA \cong \angle HED$.
3.21. Ordering of angles: (a) Exactly one of the following three conditions holds (*trichotomy*): $\angle P < \angle Q$, $\angle P \cong \angle Q$, or $\angle Q < \angle P$. (b) If $\angle P < \angle Q$ and $\angle Q \cong \angle R$, then $\angle P < \angle R$. (c) If $\angle P > \angle Q$ and $\angle Q \cong \angle R$, then $\angle P > \angle R$. (d) If $\angle P < \angle Q$ and $\angle Q < \angle R$, then $\angle P < \angle R$.
3.22. SSS test for congruence: If $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.
3.23. Euclid's fourth postulate: All right angles are congruent to each other.

Additional definition

Perpendicular: Lines ℓ and m are *perpendicular* if there is a point P that is incident with both, and points A and B that are incident with ℓ such that $A * P * B$, and points C and D that are incident with m such that $C * P * D$, such that $\angle APD$, $\angle DPB$, $\angle BPC$, and $\angle CPA$ are all right angles.

[1] Another way to phrase this is that if 2 pairs of angles, and the pair of enclosed sides, in 2 triangles are congruent, then the triangles are congruent.