Results

Theorem 4.1: Alternate interior angle theorem. In any Hilbert plane, if 2 **distinct** lines cut by a transversal have a pair of congruent alternate interior angles with respect to that transversal, then the 2 lines are parallel.

Corollary 1 to Theorem 4.1: 2 **distinct** lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point P not on line ℓ to ℓ is *unique* (and the point at which the perpendicular intersects ℓ is called its *foot*).

Corollary 2 to Theorem 4.1: If ℓ is any line and P is any point not on ℓ, there exists at least 1 line m through P parallel to ℓ.

Theorem 4.2: Exterior angle theorem. In any Hilbert plane, an exterior angle of a triangle is greater than either remote interior angle.

Corollary 1 to Theorem 4.2: If a triangle has a right or obtuse angle, the other 2 angles are acute.

Corollary 2 to Theorem 4.2: The sum of the degree measures of any 2 **distinct** angles of a triangle is less than 180°. **You may use this result only in a form that does not refer to degree measures.**

Proposition 4.1: SAA congruence criterion. Given $\overline{AC} \cong \overline{DF}$, $\sphericalangle A \cong \sphericalangle D$, and $\sphericalangle B \cong \sphericalangle E$. Then $\triangle ABC \cong \triangle DEF$.

Proposition 4.2: Hypotenuse–leg criterion. 2 right triangles are congruent if the hypotenuse and a leg of one are congruent, respectively, to the hypotenuse and a leg of the other.

Proposition 4.3: Every segment has a unique midpoint.

Proposition 4.4:

(a) Every angle has a unique bisector.
(b) Every segment has a unique perpendicular bisector.

Proposition 4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle; i.e., $\overline{AB} > \overline{BC}$ if and only if $\sphericalangle C > \sphericalangle A$.

Proposition 4.6: Given $\triangle ABC$ and $\triangle A'B'C'$, if we have $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\sphericalangle B < \sphericalangle B'$ if and only if $\overline{AC} < \overline{A'C'}$.

Triangle inequality: If AB, BC, and AC are the lengths of the sides of a triangle $\triangle ABC$, then $AC < AB + BC$. **You may use this result only in a form that does not refer to segment lengths.**

Equivalents of the Euclidean parallel postulate

The *Euclidean parallel postulate* says:

> For any line ℓ, and any point P that is not incident with ℓ, there exists exactly 1 line m that is incident with P and parallel to ℓ.

By Corollary 1 to Theorem 4.1 (which is the same statement, with 'exactly' replaced by 'at least'), we could say 'at most' instead of 'exactly'. The text calls this apparently weaker version *Hilbert's Euclidean parallel postulate.*

The following results (except for the last) are equivalent to the Euclidean parallel postulate. The last result is implied by, but does not imply, the Euclidean parallel postulate. You **must not** use any of them unless you are explicitly allowed to assume that postulate.

Theorem 4.4: If 2 lines are intersected by a transversal in such a way that the sum of the degree measures of the 2 interior angles on one side of the transversal is less than 180°, then the 2 lines meet on that side of the transversal. **You may use this result only in a form that does not refer to degree measures.**

Proposition 4.7: If a line intersects one of 2 parallel lines, then it also intersects the other.

Exercise 4.10: Transitivity of parallelism. **If 2 distinct lines are both parallel to a given line, then they are parallel to each other.**

Proposition 4.8: If 2 parallel lines are cut by a transversal, then the pairs of alternate interior angles that are formed are congruent.

Proposition 4.9: If t is a transversal to ℓ and m, $\ell \parallel m$, and $t \perp \ell$, then $t \perp m$.

Proposition 4.10: If $k \parallel \ell$, $m \perp k$, and $n \perp \ell$, then either $m = n$ or $m \parallel n$.

Proposition 4.11: For every triangle $\triangle ABC$, $(\sphericalangle A)^\circ + (\sphericalangle B)^\circ + (\sphericalangle C)^\circ = 180^\circ$. **You may use this result only in a form that does not refer to degree measures.**

Definitions

Circle, centre, radius: Given distinct points O and A. The set of all points P such that segment $\overline{OP}$ is congruent to segment $\overline{OA}$ is called *the circle with O as centre and $\overline{OA}$ as radius.* For each point P in that set, we say that P *lies on* the circle and $\overline{OP}$ is called a *radius* of the circle.

Opposite rays: Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are *opposite* if they are distinct, if they emanate from the same point A, and if they are part of the same line $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.

Angle, sides, coterminal: An *angle with vertex A* is a point A together with 2 distinct, non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the *sides* of the angle) emanating from A. If $r = \overrightarrow{AB}$ and $s = \overrightarrow{AC}$, then rays r, s are said to be *coterminal*, and the angle is also denoted $\sphericalangle(r, s)$.

Supplement: If 2 angles $\sphericalangle DAB$ and $\sphericalangle CAD$ have a common side $\overrightarrow{AD}$ and the other 2 sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, the angles are *supplements* of each other, or *supplementary angles.* (Notice that an angle has 2 supplements.)

Right angle: An angle $\sphericalangle BAD$ is a *right angle* if it has a supplementary angle to which it is congruent.

Parallel: 2 lines ℓ and m are *parallel* if they do not intersect, i.e., if no point lies on both of them. We denote this by $\ell \parallel m$.

Collinear: 3 or more points A, B, C, . . . are *collinear* if there exists a line incident with all of them.

Concurrent: 3 or more lines ℓ, m, n, . . . are *concurrent* if there exists a point incident with all of them.

Equilateral triangle: A triangle in which all pairs of sides are congruent.

Isosceles triangle, base, base angle, vertex angle: A triangle in which 2 pairs of sides are congruent. If A, B, and C are 3 non-collinear points and $\overline{AB} \cong \overline{AC}$, then we say that $\overline{BC}$ is a *base* of $\triangle ABC$, $\sphericalangle B$ and $\sphericalangle C$ are *base angles*, and $\sphericalangle A$ is a *vertex angle*, of $\triangle ABC$. (In an isosceles triangle that is also equilateral, all sides are bases, and all angles are both vertex angles and base angles.)

Hilbert plane: A model of Axioms I1–I3, B1–B4, and C1–C6. The Euclidean parallel postulate *need not* hold in a Hilbert plane.

Transversal: If ℓ, m, and t are 3 lines, then t is a *transversal* to ℓ and m if it is parallel to neither ℓ nor m, and the lines ℓ, m, and t are not concurrent.

(Alternate) interior angles: Suppose that t is a transversal to ℓ and ℓ', with t meeting ℓ at B and ℓ' at B', and A is a point on ℓ and C' is a point on ℓ' such that A **and** C' **are on opposite sides of** t, then $\sphericalangle ABB'$ and $\sphericalangle BB'C$ are *(alternate) interior angles.*

Exterior angle, remote interior angle: Suppose that A, B, and C are 3 non-collinear points and D is a point such that $B * C * D$, then $\sphericalangle ACD$ is an *exterior angle* to $\triangle ABC$. Its *remote interior angles* are $\sphericalangle ABC$ and $\sphericalangle BAC$.

Continuity principles

Inside, outside: If $\mathscr{C}$ is a circle with centre O, and C is a point that is incident with $\mathscr{C}$, then a point P is *inside* $\mathscr{C}$ if and only if $P = O$, or $\overline{OP}$ is shorter than $\overline{OC}$; and *outside* $\mathscr{C}$ if and only if $\overline{OP}$ is longer than $\overline{OC}$.

Circle–circle continuity principle: If a circle $\mathscr{C}$ has 1 point inside and 1 point outside another circle $\mathscr{C}'$, then the 2 circles intersect in 2 **distinct** points.

Line–circle continuity principle: If a line passes through a point inside a circle, then the line intersects the circle in 2 **distinct** points.

Segment–circle continuity principle: If one endpoint of a segment is inside a circle and the other endpoint is outside, then the segment intersects the circle at a point in between.