1. 3 September

Definition 1.1. An ℓ-group is a Hausdorff topological group G such that G has a neighbourhood basis at the identity consisting of compact, open subgroups. For such a group G, we write $C_c^\infty(G)$ for the space of functions $f : G \to \mathbb{C}$ such that f is compactly supported and locally constant. (This means that, for each element $g \in G$, there is an open subgroup K_g of G such that $f(gk) = f(g)$ for all $k \in K_g$.) We will sometimes say that a function is smooth instead of locally constant. (See Definition 2.4 for the motivation.)

Exercise 1.1. Prove that a compactly supported function is smooth in the above sense if and only if there is an open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$ and $g \in G$.

Definition 1.2. If G is an ℓ-group and K is a compact open subgroup of G, then we define the left Haar measure G, normalised with respect to K, to be the function $\int_G \cdot(g)dg : C_c^\infty(G) \to \mathbb{C}$ defined as follows. For $f \in C_c^\infty(G)$, let K' be a compact open subgroup of G, contained in K, such that f is invariant under right translation by K. Then $\int_G f(g)dg := [K : K']^{-1} \sum_{g \in G/K'} f(g)$. If $G = K$, then we will simply call this measure the normalised Haar measure on K.

Exercise 1.2′. Prove that $\int_G f(g)dg$ is independent of the subgroup K' chosen above, and that $f \mapsto \int_G f(g)dg$ is a linear functional on $C_c^\infty(G)$.

Exercise 1.2. With the above notation, $f \mapsto \int_G f(gh)dg$ is a left Haar measure normalised with respect to K^h, hence is a rational multiple $\delta_G(h)$ of the Haar measure normalised with respect to K. Show that $f \mapsto \int_G f(g)\delta_G(g)dg$ is (with the obvious terminology) a right Haar measure on G, normalised with respect to K. Conclude that, if $G = K$ is compact, then every left Haar measure is also a right Haar measure.

Definition 1.3. In the terminology of Definitions 2.3 and 2.4, δ_G is a smooth character of G. We call it the modulus character of G. If δ_G is trivial, then we say that G is unimodular.

Definition 1.4. A p-adic group is a first-countable ℓ-group G such that G/K is countable for all compact, open subgroups K of G.

Exercise 1.3. Check that k, $k^\times$, and the various $\mathrm{GL}_N(k)$ are all p-adic groups. (Hint: For k, take the compact, open subgroups to be powers of the prime ideal P. For $k^\times$, take elements that differ by 1 from an element of a large power of the prime ideal P. For $\mathrm{GL}_N(k)$, constrain matrix entries appropriately.)

More generally, show that, if H is any separable group and U is an open subgroup, then H/U is countable. Verify that k, $k^\times$, and the various $\mathrm{GL}_N(k)$ are separable.

2. 5 September

Definition 2.1. A representation of G (or G-representation) is a homomorphism π from G to $\mathrm{GL}(V)$, for some complex vector space V. We will say that V, or π, or (π, V) is a representation of G, or that (G, π) is a representation, omitting ingredients as convenient. We will sometimes also call V a G-module, and write gv instead of $\pi(g)v$ for $g \in G$ and $v \in V$.

Example 2.2. If V is the space of all complex-valued functions, or of all compactly supported, complex-valued functions, on G, then we can define maps $L : G \to \mathrm{GL}(V)$ and $R : G \to \mathrm{GL}(V)$ by $L(g)f : g' \mapsto f(g^{-1}g')$ and $R(g)f : g' \mapsto f(g'g)$ for all $g \in G$ and $f \in V$. These are (restrictions of) the LEFT (respectively, RIGHT) REGULAR REPRESENTATIONS of G.

Definition 2.3. If V is 1-dimensional, then π is essentially a homomorphism from G into $\mathbb{C}^\times$. We call such a homomorphism a CHARACTER, or, if the need arises to distinguish it from the character of a larger-dimensional representation, a LINEAR CHARACTER. (Some authors use QUASI-CHARACTER instead, and reserve 'character' for the unitary case.)

Definition 2.5. If (π', V') is another representation of G, then we will write $\mathrm{Hom}_G(V, V')$ or $\mathrm{Hom}_G(\pi, \pi')$ (or even $\mathrm{Hom}_G(V, \pi')$ or $\mathrm{Hom}_G(\pi, V')$) for the set of linear maps $T : V \to V'$ such that $T(\pi(g)v) = \pi'(g)Tv$ for all $v \in V$ and $g \in G$. If $V = V'$, then we will write just $\mathrm{End}_G(V)$. This latter space is actually an algebra, with composition as multiplication; we will call it the COMMUTING ALGEBRA or INTERTWINING ALGEBRA of V. We will call the elements of $\mathrm{Hom}_G(V, V')$ G-MAPS. We say that V and V' are EQUIVALENT, and write $V \cong V'$, if and only if there is an invertible G-map between them. (Of course, one need only check invertibility as a map of linear spaces.)

Definition 2.4. If K is a subgroup of G, then we write V^K for the set of vectors in V fixed by every element of K. A vector $v \in V$ is said to be SMOOTH if and only if there is some open subgroup K of G such that $v \in V^K$. The set of smooth vectors in V is denoted by V^∞. (EXERCISE 2.1: Verify that V^∞ is a G-submodule of V.) We say that V is SMOOTH if and only if $V = V^\infty$. The collection of smooth representations of G forms an Abelian category, which we denote by $\mathfrak{R}(G)$, in the natural way.

Exercise 2.2. Suppose that V is finite dimensional. Show that it is smooth if and only if the map $\pi : G \to \mathrm{GL}(V)$ is continuous for the usual topology on the complex group $\mathrm{GL}(V)$.

Definition 2.6. If K is a compact subgroup of G and $v \in V$, then we define $e_K v = \int_K kv\,dk$ for $v \in V$, where dk is normalised Haar measure on K.

Lemma 2.7. *A sequence* $V' \xrightarrow{i} V \xrightarrow{j} V''$ *of smooth representations is exact if and only if the sequence* $V'^K \xrightarrow{i^K} V^K \xrightarrow{j^K} V''^K$ *is exact for all compact open subgroups* K *of* G.

Definition 2.8. The representation V is said to be REDUCIBLE if and only if there is a proper, non-0 G-submodule of V. It is IRREDUCIBLE if and only if it is not 0 and not reducible. (Note that we write 0 for the 0 space, as well as the 0 vector.) The set of equivalence classes of (smooth) irreducible representations of G is denoted by $\Pi(G)$. We will often confuse an irreducible representation with its equivalence class.

Lemma 2.9 (Schur's lemma)**.** *If* V *is irreducible, then the natural embedding of* $\mathbb{C}$ *in* $\mathrm{End}_G(V)$ *is an isomorphism.*

Lemma 2.10. *If* V *is irreducible, then* V *has countable dimension.*

3. 8 September

Exercise 3.1. Give an example of a group G and a reducible G-module V such that $\mathrm{End}_G(V)$ is 1-dimensional. (Hint: Note that V must be indecomposable; that is, it must not be able to be written as the direct sum of two proper G-submodules.)

Corollary 3.2. *If V and V' are both irreducible representations of G, then $\mathrm{Hom}_G(V, V') = 0$ unless V and V' are equivalent, in which case $\mathrm{Hom}_G(V, V')$ is 1-dimensional.*

Definition 3.3. We say that V admits a central character if and only if there is a function ω_V from $Z(G)$ to $\mathbb{C}$ such that $z \cdot v = \omega_V(z)v$ for all $z \in Z(G)$ and $v \in V$. In this case, ω_V is the unique such function (as long as $V \neq 0$), and it is a character. We call it the central character of V.

Corollary 3.4. *If V is irreducible, then it admits a central character.*

Definition 3.5. The representation V is semisimple if and only if there is a set $\{V_i : i \in I\}$ of irreducible G-submodules of V such that $V = \sum_{i \in I} V_i$. (Notice that the set may be empty; that distinct elements of the set may be equivalent; and that we are not requiring that the sum be direct — but see Exercise 3.7 for this last point.)

Exercise 3.7. Show that a representation is semisimple if and only if it can be written as a *direct* sum of irreducible submodules.

Lemma 3.8. *The representation V is semisimple if and only if, for every G-submodule V' of V, there exists a G-submodule V'' of V such that $V = V' \oplus V''$.*

Exercise 3.9. Give an example of a group G and a G-module that has no irreducible G-submodule.

Lemma 3.10. *If V is non-0 and finitely generated (as a G-module), then it has an irreducible quotient.*

Corollary 3.11. *Every non-0 G-module has an irreducible G-subquotient.*

Example 3.6. The homomorphism $t \mapsto \begin{pmatrix} 1 & \operatorname{ord} t \\ 0 & 1 \end{pmatrix}$ from k to $\mathrm{GL}_2(k)$ specifies a 2-dimensional representation of k that is not semisimple.

Definition 3.12. The representation V is said to be admissible if and only if, for every compact, open subgroup K of G, we have that V^K is finite dimensional.

Exercise 3.13. Find a group G and a (smooth) irreducible representation of G that is not admissible. (I do not know of one.)

4. 10 September

Definition 4.1. We say that V is unitary (or unitarisable) if there is a G-invariant, positive-definite, Hermitian form $\langle \cdot, \cdot \rangle$ on V.

Lemma 4.2. *Suppose that V is admissible and unitarisable, and $V' \subseteq V$ is a G-submodule. Then V is semisimple.*

Lemma 4.3. *If G is compact, then V is semisimple.*

Exercise 4.3′. Prove this.

Lemma 4.4. *If G is compact, then every open subgroup contains a normal, open subgroup; and every irreducible G-representation is finite dimensional, and factors through the quotient by some normal, open subgroup.*

Lemma 4.5. *If $\operatorname{End}_G(V)$ is 1-dimensional and V is semisimple (in particular, if G is compact, or V is admissible and unitarisable), then V is irreducible.*

5. 12 September

Definition 5.1. If H is a closed subgroup of G and (σ, W) is an irreducible representation of H, then we define the MULTIPLICITY of (H, σ) in V to be $\dim_{\mathbb{C}} \operatorname{Hom}_H(W, \operatorname{Res}^G_H V)$, and the (H, σ)-ISOTYPIC COMPONENT $V^{(H,\sigma)}$ of V to be the image of the evaluation map $\operatorname{Hom}_H(W, \operatorname{Res}^G_H V) \otimes_{\mathbb{C}} W$.

Exercise 5.2. Show that, with the notation of Definition 5.1, the restriction of the action of G makes $V^{(H,\sigma)}$ an H-module that is equivalent to the direct sum of $m_{(H,\sigma)}(V)$ copies of σ; in particular, is semisimple. Conclude that V is semisimple if and only if it is the sum of its π-isotypic components for $\pi \in \Pi(G)$. (HINT: Use Schur's lemma.)

Lemma 5.3. *The representation V is admissible if and only if, for some (equivalently, every) compact, open subgroup K of G, we have that the multiplicity of every irreducible representation of K in V is finite.*

Definition 5.4. The algebraic dual V^* of V carries a natural structure of a G-representation, namely, $gv^* : v \mapsto v^*(g^{-1}v)$ for all $g \in G$ and $v^* \in V^*$. We write $\widetilde{V}$ for the space $(V^*)^\infty$ of smooth vectors in this representation. This is called the CONTRAGREDIENT of V. If $v \in V$ and $\tilde{v} \in \widetilde{V}$, then we will often write $\langle \tilde{v}, v \rangle$ in place of $\tilde{v}(v)$.

Lemma 5.5. *The representation V is admissible if and only if the natural map $V \to \widetilde{\widetilde{V}}$ is an isomorphism.*

Lemma 5.6. *For any compact open subgroup K of G, the restriction map $\widetilde{V}^K \to (V^K)^*$ is an isomorphism.*

(EXERCISE 5.7: Check that e_K^* is a projection onto $\widetilde{V}^K$.)

Definition 5.8. We denote by $\mathcal{H} = \mathcal{H}_n$ the group of all $(n+2)$-square matrices of the form

$$[v_+ \oplus v_-; z] := \begin{pmatrix} 1 & v_+^{\mathrm{t}} & z + \frac{1}{2}(v_+, v_-) \\ 0 & I_n & v_- \\ 0 & 0 & 1 \end{pmatrix},$$

where $v_+, v_- \in k^n$ are viewed as column vectors; $z \in k$; and $(\cdot, \cdot)$ is the standard inner product on k^n. The group operation is matrix multiplication. Then $\mathcal{H}$ is a HEISENBERG GROUP.

$$[v; z] \cdot [v'; z'] = \left[v + v'; z + z' + \tfrac{1}{2}\langle v, v' \rangle\right], \tag{5.9}$$

$$\left[[v; z], [v'; z']\right] = [0; \langle v, v' \rangle] \tag{5.10}$$

6. 15 September

Lemma 6.1. *If V is an irreducible representation of $\mathcal{H}$ with trivial central character, then V is 1-dimensional.*

Corollary 6.2. *An irreducible representation of $\mathcal{H}$ with trivial central character is admissible.*

Definition 6.3. Suppose that H is a closed subgroup of G, and (σ, W) is a representation of H. Consider the vector space V of all functions $f : G \to W$ such that $f(hg) = \sigma(h)f(g)$ for all $g \in G$ and $h \in H$. Then G acts by V on right translation. Specifically, for $f \in V$ and $g_0 \in G$, we write $\pi(g)f$ for the function $g \mapsto f(gg_0)$ from G to W. Then $\pi(g)f \in V$. The representation (π, V^∞) of G is called the (SMOOTH) INDUCED REPRESENTATION from (σ, W), and denoted by $(\operatorname{Ind}_H^G \sigma, \operatorname{Ind}_H^G W)$. (We will later have occasion to call this process UN-NORMALISED INDUCTION.)

Definition 6.4. Suppose that (π, V) is a representation of G, and θ is a character of G. Then $(\pi \otimes \theta, V_\theta)$ is the representation of G for which $V_\theta = V$ as vector spaces, but $(\pi \otimes \theta)(g)v = \theta(g) \cdot \pi(g)v$ for all $g \in G$ and $v \in V$. We call this representation the (CENTRAL) TWIST OF π BY θ.

Exercise 6.5. Preserve the notation of Definition 6.3.

(1) Suppose that H contains $Z(G)$ and W has central character ω_W. Prove that $\operatorname{Ind}_H^G W$ has central character $\omega_W\big|_{Z(G)}$.
(2) Prove that $\operatorname{Ind}_H^G W_{\theta\big|_H}$ is equivalent to $(\operatorname{Ind}_H^G W)_\theta$ for all characters θ of G.

Definition 6.6. A pair $(\widehat{\mathcal{W}}_+, \widehat{\mathcal{W}}_-)$ of closed, Abelian subgroups of $\mathcal{H}$ is said to be a COMPLETE POLARISATION of $\mathcal{H}$ (with respect to χ) if the following properties hold.

(1) The multiplication map $\widehat{\mathcal{W}}_+ \times Z(\mathcal{H}) \times \widehat{\mathcal{W}}_- \to \mathcal{H}$ is a homeomorphism.
(2) The pairing $(\hat{w}_+, \hat{w}_-) \mapsto \chi[\hat{w}_-, \hat{w}_+]$ puts the groups $\widehat{\mathcal{W}}_+$ and $\widehat{\mathcal{W}}_-$ in topological duality. (This means that, if we equip the group $\widehat{\mathcal{W}}_+^* := \operatorname{Hom}_{\mathrm{sm}}(\widehat{\mathcal{W}}_+, \mathbb{C})$ of (smooth) characters of $\widehat{\mathcal{W}}_+$ with the compact-open topology, then the pairing affords a topological isomorphism $\widehat{\mathcal{W}}_- \to \widehat{\mathcal{W}}_+^*$; and similarly with the rôles of $\widehat{\mathcal{W}}_-$ and $\widehat{\mathcal{W}}_+$ reversed.)

Definition 6.7. We say that a p-adic group G is EXHAUSTED BY ITS COMPACT, OPEN SUBGROUPS if every compact subset of G is contained in a compact, open subgroup of G.

Exercise 6.8. Check that

$$\left(\widehat{\mathcal{W}}_+ := \{[w_+ \oplus 0; 0] : w_+ \in k^n\}, \widehat{\mathcal{W}}_- = \{[0 \oplus w_-; 0] : w_- \in k^n\}\right)$$

is a complete polarisation of $\mathcal{H}$.

Exercise 6.9. Show that any character of a group that is exhausted by its compact, open subgroups is unitary; and that $\widehat{\mathcal{W}}_+$, $\widehat{\mathcal{W}}_-$, and $\widetilde{\mathcal{W}}_+ := \widehat{\mathcal{W}}_+ Z(\mathcal{H})$ are exhausted by their compact, open subgroups.

7. 17 September

Lemma 7.1. *The action of* $\widehat{\mathcal{W}}_-$ *on* X_χ *is simply transitive.*

Theorem 7.2 (Frobenius reciprocity). *If* H *is a closed subgroup of* G *and* W *is a representation of* H*, then the map* $T \mapsto \big(v \mapsto (Tv)(1)\big)$ *is an isomorphism from* $\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$ *to* $\operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$.

(Exercise 7.3: Check that $S \in \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$. (Hint: This requires precisely the transformation properties that Tv has.))

Theorem 7.3. *The representation* $\omega_\chi := \operatorname{Ind}_{\widetilde{\mathcal{W}}_+}^{\mathcal{H}} \tilde{\chi}$ *is irreducible and has central character* χ*.*

Lemma 7.4. ω_χ *is admissible.*

Corollary 7.5. *Restriction to* $\widehat{\mathcal{W}}_-$ *specifies a vector space isomorphism* $\operatorname{Res}_{\widetilde{\mathcal{W}}_+}^{\mathcal{H}} \omega_\chi \to C_c^\infty(\widehat{\mathcal{W}}_-)$.

Exercise 7.6. Prove Corollary 7.5.

8. 19 September

Exercise 8.1. Show that the equivalence class of ω_χ is independent of the choice of $\tilde{\chi} \in X_\chi$. (Hint: Use Lemma 7.1.)

Lemma 8.2. ω_χ *is unitary.*

Lemma 8.3. *The intertwining algebra* $\operatorname{End}_{\mathcal{H}}(\omega_\chi)$ *is* 1*-dimensional.*

(Exercise 8.4: Show that the smooth characters of an Abelian group separate points.)

Definition 8.5. Suppose that G is a p-adic group, H is a closed subgroup, V is a representation of G, and θ is a character of H. Put $V(H, \theta) = \operatorname{Span}\{hv - \theta(h)v : h \in H, v \in V\}$ and $V_{(H,\theta)}$. Then $V_{(H,\theta)}$ is called the space of (H, θ)-coinvariants for V. It is the largest quotient of $\operatorname{Res}_H^G V$ that is θ-isotypic.

Exercise 8.6. With the notation of Definition 8.5, suppose that H is exhausted by its compact, open subgroups. Show that $v \in V(H, \theta)$ if and only if there is some compact, open subgroup K of H such that $\int_K \theta(k)^{-1}(h \cdot k)dk = 0$ for any Haar measure dk on K. Show also that the integral still vanishes if we enlarge K.

Exercise 8.7. With the notation of Definition 8.5, suppose that H is compact. Show that $V^{(H,\theta)} \cong V_{(H,\theta)}$. (Hint: Consider the integral of Exercise 8.6. Use it to define a suitable analogue of the operator e_K of Definition 2.6.)

9. 22 September

Theorem 9.1. *Any representation of* $\mathcal{H}$ *with central character* χ *admits* ω_χ *as a quotient.*

Corollary 9.2 (p-adic Stone–von Neumann theorem). *There is a unique equivalence class of irreducible representations of* $\mathcal{H}$ *with central character* χ*.*

Corollary 9.3. *All irreducible representations of* $\mathcal{H}$ *are admissible.*

Definition 9.4. Write $\widetilde{\mathbb{R}}$ for the disjoint union of $\mathbb{R}$ with $\{\infty\}$ and a set of symbols $\{r+ : r \in \mathbb{R}\}$. For $r \in \mathbb{R}$, put

$$Z(\mathcal{H})_r := \{[0; z] : z \in 1 + P^{\lceil r \rceil}\}$$

and

$$Z(\mathcal{H})_{r+} := \{[0; z] : z \in 1 + P^{\lceil r+ \rceil}\} = \bigcup_{t > r} Z(\mathcal{H})_t,$$

where $\lceil r \rceil$ is the smallest integer that is no less than r, and $\lceil r+ \rceil$ is the smallest integer that is greater than r. Put $Z(\mathcal{H})_\infty = \{1\}$.

Definition 9.5. The DEPTH $\mathrm{d}(\chi)$ of χ is the least number $r \in \mathbb{R}$ such that $Z(\mathcal{H})_{r+} \subseteq \ker \chi$.

Definition 9.6. A LATTICE in a k-vector space is a full-rank $\mathcal{O}$-submodule of that vector space. The DUAL of a lattice $\Lambda \subseteq \mathcal{W}$ is

$$\Lambda^\perp = \{w \in \mathcal{W} : \langle w, \lambda \rangle \in \mathcal{O} \text{ for all } \lambda \in \Lambda\}.$$

Let Λ be a self-dual lattice in $\mathcal{W}$, so that $\Lambda = \Lambda^\perp$. For $r \in \mathbb{R}$, put

$$J_{\Lambda,r} = \{[w; z] : w \in P^{\lceil r/2 \rceil}\Lambda \text{ and } z \in P^{\lceil r \rceil}\}$$

and

$$J_{\Lambda,r+} = \{[w; z] : w \in P^{\lceil (r/2)+ \rceil}\Lambda \text{ and } z \in P^{\lceil r+ \rceil}\} = \bigcup_{t > r} J_{\Lambda,t}.$$

Put $J_{\Lambda,\infty} = 1$.

Example 9.7. If $\{w_{i+} : i \in I\} \cup \{w_{i-} : i \in I\}$ is a symplectic basis of $\mathcal{W}$, so that $\mathrm{Span}\{w_{i+} : i \in I\}$ and $\mathrm{Span}\{w_{i-} : i \in I\}$ are totally isotropic and $\langle w_{i+}, w_{j-} \rangle = \delta_{ij}$ for $i, j \in I$, then $\bigoplus_{i \in I} \mathcal{O}w_{i+} \oplus \mathcal{O}w_{i-}$ is a self-dual lattice in $\mathcal{W}$. In fact, every self-dual lattice arises in this way, for a suitable choice of basis.

10. 24 September

Definition 10.1. Put $\mathcal{W} = J_{\Lambda,\mathrm{d}(\chi)}/Z(\mathcal{H})_{\mathrm{d}(\chi)} J_{\Lambda,\mathrm{d}(\chi)+} \cong P^{\lceil \mathrm{d}(\chi) \rceil}\Lambda / P^{\lceil \mathrm{d}(\chi)+ \rceil}\Lambda$ — a finite, Abelian group in which every element has order dividing p, hence carries a natural structure of an $\mathbb{F}_p$-vector space.

Exercise 10.2. Show that the commutator map $\mathcal{W} \times \mathcal{W} \to Z(\mathcal{H})_{\mathrm{d}(\chi)} / \ker \chi\big|_{Z(\mathcal{H})_{\mathrm{d}(\chi)}}$ is a non-degenerate symplectic form on $\mathcal{W}$.

Exercise 10.2. Let $\mathcal{W}$ be as in Definition 10.1 and $\mathcal{W}^\sharp$ as in Remark **??**, so that we have an exact sequence

$$1 \to \mathbb{F}_p \to \mathcal{W}^\sharp \to \mathcal{W} \to 1.$$

Note that we also have the exact sequence

$$1 \to \mathbb{F}_p \cong Z(\mathcal{H})_{\mathrm{d}(\chi)} / \ker \chi\big|_{Z(\mathcal{H})_{\mathrm{d}(\chi)}} \to J_{\Lambda,\mathrm{d}(\chi)} / \ker \chi\big|_{J_{\Lambda,\mathrm{d}(\chi)}} \to \mathcal{W} \to 1.$$

Show that these two extensions of $\mathcal{W}$ by $\mathbb{F}_p$ are isomorphic. (See Lemma 10.1 of [`?yu:supercuspidal`].)

Exercise 10.3. Suppose that G is a p-adic group, and V and V' are representations of G. Show that there is a natural injection of $\mathrm{Hom}_G(V, V')$ into $\mathrm{Hom}_G(\widetilde{V}', \widetilde{V})$. Under what conditions on V and V' is this map an isomorphism?

Exercise 10.4. Suppose that G is a p-adic group, and V and V' are two representations of G. Show that $\mathrm{Hom}_G(V, \widetilde{V'})$ is naturally isomorphic to the space of G-invariant pairings $V \times V' \to \mathbb{C}$. Conclude that it is naturally isomorphic to $\mathrm{Hom}_G(V', \widetilde{V})$.

Definition 10.5. In the setting of Definition 6.3, we write $\text{c-Ind}_H^G W$ for the space of all functions $f \in \mathrm{Ind}_H^G W$ such that there is a compact subset C of G with the property that $\operatorname{supp} f \subseteq HC$. This is a G-submodule of $\mathrm{Ind}_H^G W$. It is called the the COMPACTLY INDUCED REPRESENTATION from (σ, W).

11. 26 September

Definition 11.1. If G is a p-adic group, H is a closed subgroup, and θ is a character of H, then a θ-INVARIANT MEASURE on G, relative to H, is a non-0, positive, G-invariant functional on $\text{c-Ind}_H^G \theta$. (Here, we say that a functional is "positive" if it takes functions that are non-negative real valued to non-negative real numbers.)

Lemma 11.2. *With the notation of Definition 11.1, there exists a θ-invariant measure on G, relative to H, if and only if $\theta = \delta_{H\backslash G} = \delta_H^{-1}\delta_G\big|_H$; and such a measure is unique up to multiplication by a positive, real number.*

Exercise 11.3. Prove Lemma 11.2. (See [?bushnell-henniart:llc-gl2, §3.4].)

Lemma 11.3. *With the notation of Definition 10.5, suppose that H is open and V is a representation of G. Then the map*

$$S \mapsto \big(f \mapsto \sum_{g \in G/H} g^{-1} \cdot S(f(g))\big)$$

is an isomorphism from $\mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$ *to* $\mathrm{Hom}_G(\text{c-Ind}_H^G W, V)$.

(EXERCISE 11.4: Verify that the function $\hat{w} : G \to W$ that is 0 outside of H, and restricts to $h \mapsto hw$ on H, lies in $\text{c-Ind}_H^G W$.)

Corollary 11.5. *With the notation and hypotheses of Lemma 11.3, there exists a θ-invariant measure on G, relative to H, if and only if $\theta = 1$.*

Theorem 11.6. *In the notation of Definition 10.5, let $d(h\backslash g)$ be a $\delta_{H\backslash G}$-invariant measure on G, relative to H. Then there is a canonical isomorphism between* $\mathrm{Ind}_H^G \widetilde{W}_{\delta_{H\backslash G}}$ *and* $(\text{c-Ind}_H^G W)^\sim$.

Exercise 11.7. Show that $\mathcal{H}$ is unimodular. (HINT: Show that the modulus character factors through a character of an Abelian quotient of $\mathcal{H}$.)

12. 29 September

Lemma 12.1. *Let G be a p-adic group and $V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$ an exact sequence of G-modules. Then $0 \to \widetilde{V''} \xrightarrow{\tilde{j}} \widetilde{V} \xrightarrow{\tilde{\imath}} \widetilde{V'}$ is exact.*

Lemma 12.2. *If G is a p-adic group and V is an irreducible, admissible representation of G, then $\widetilde{V}$ is also irreducible.*

Lemma 12.3. $\mathrm{Ind}_{Z(\mathcal{H})J_{\Lambda,\mathrm{d}(\chi)}}^{\mathcal{H}} \omega_{\overline{\chi}} = \text{c-Ind}_{Z(\mathcal{H})J_{\Lambda,\mathrm{d}(\chi)}}^{\mathcal{H}} \omega_{\overline{\chi}}$.

Proposition 12.4. *The representation* $\mathrm{Ind}_{J_{\Lambda,\mathrm{d}(\chi)}}^{\mathcal{H}} \omega_{\overline{\chi}}$ *is isomorphic to* ω_χ.

(EXERCISE 12.5: Modify the proof of Theorem 9.1 to show that there is an $\mathcal{H}$-module map $\omega'_\chi \to \omega_\chi$ that is not 0 on V''_χ.)

13. 1 October

Definition 13.1. If G is a p-adic group, V is a representation of G, $v \in V$, and $\tilde{v} \in \widetilde{V}$, then we define the MATRIX COEFFICIENT $m_{\tilde{v},v}$ to be the function from G to $\mathbb{C}$ given by $g \mapsto \langle \tilde{v}, gv \rangle$.

Definition 13.2. If G is a p-adic group and V is a representation of G, then V is a DISCRETE-SERIES representation if and only if it admits a unitary central character and, for all $v \in V$ and $\tilde{v} \in \widetilde{V}$, we have that $|m_{\tilde{v},v}| \in L^2(Z(G)\backslash G)$ (with respect to any Haar measure on $Z(G)\backslash G$); and V is an ESSENTIALLY DISCRETE-SERIES representation if and only if there is a character θ of G such that V_θ is a discrete-series representation.

Definition 13.3. With the notation of Definition 13.2, we say that V is SUPERCUSPIDAL if and only if, for all $v \in V$ and $\tilde{v} \in \widetilde{V}$, the image in $Z(G)\backslash G$ of the support of $m_{\tilde{v},v}$ is compact.

Exercise 13.4. With the notation of Definitions 13.2 and 13.3, suppose that V is irreducible and admits a unitary central character. Then V is a discrete series (respectively, supercuspidal) representation if and only if there exist $v \in V \setminus \{0\}$ and $\tilde{v} \in \widetilde{V} \setminus \{0\}$ such that $|m_{\tilde{v},v}|$ lies in $L^2(Z(G)\backslash G)$ (respectively, is compactly supported).

Lemma 13.5. *Every irreducible representation of a unimodular p-adic group G that arises by compact induction from an open, compact-modulo-centre subgroup of G is supercuspidal.*

14. 8 October

Lemma 14.1. *Every irreducible representation of a unimodular p-adic group G that arises by compact induction from an open, compact-modulo-centre subgroup of G is supercuspidal.*

Corollary 14.2. *Every irreducible representation of $\mathcal{H}$ with non-trivial central character is supercuspidal, and arises by induction from some open, compact-modulo-centre subgroup of $\mathcal{H}$.*

Theorem 14.3. *If G is a p-adic group and V is an irreducible, supercuspidal representation of G, then V is admissible.*

(EXERCISE 14.4: Show that S injects into $Z(G)K\backslash G$, and that $\langle v^*, e_K w \rangle = \langle \tilde{v}, w \rangle$ for all $w \in V$. (HINT: This latter is essentially Exercise 5.7.))

Definition 14.5. A PROJECTIVE REPRESENTATION of a p-adic group G is a homomorphism $\overline{\pi} : G \to \mathrm{PGL}(V)$ for some complex vector space V. A LIFTING of the projective representation $\overline{\pi}$ is a homomorphism $\pi : G \to \mathrm{GL}(V)$ such that the composition of π with the natural homomorphism $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ is $\overline{\pi}$.

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Exercise 16.1. Show that, if $\mathcal{W}$ is 2-dimensional, then $\mathrm{Sp}(\mathcal{W}) \cong \mathrm{SL}(\mathcal{W})$ (i.e., $\mathrm{Sp}_2(k) \cong \mathrm{SL}_2(k)$).

Definition 16.2. Let P be the subgroup of G consisting of those elements $g \in G$ such that $g\mathcal{W}_+ = \mathcal{W}_+$. This is called a SIEGEL PARABOLIC SUBGROUP of G. (The general notion of a parabolic subgroup will appear in Definition **??**.)

Exercise 16.3. Preserve the notation of Remark **??**. Show that P consists of those elements $p = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$ for which $A_{++}A_{--}^* = 1$, $A_{-+} = 0$, and $A_{++}A_{+-}^* = A_{+-}A_{++}^*$.

Exercise 16.4. Show that the map that sends a right coset gP of P in G to $g\mathcal{W}_+$ is a bijection from G/P to the set of Lagrangians (i.e., maximal totally isotropic subspaces) of $\mathcal{W}$.

Exercise 16.5. Let L be the group of matrices $g \in P$ such that $g\mathcal{W}_- = \mathcal{W}_-$, and U the group of matrices $g \in P$ such that g fixes $\mathcal{W}_+$ pointwise. Show that $L \cong \mathrm{GL}_k(\mathcal{W}_+)$ and $P = L \ltimes U$. This semi-direct product decomposition is called a LEVI DECOMPOSITION. The group L is called a LEVI COMPONENT of P, and the group U is the UNIPOTENT RADICAL of P.

Exercise 16.6. Show that, if G' is a p-adic group, H' is a closed subgroup, and W is a representation of H', then the evaluation map $\text{c-Ind}_{H'}^{G'} W \to W$ sending $f \in \text{c-Ind}_{H'}^{G'} W$ to $f(1)$ is surjective. Conclude that $\text{c-Ind}_{H'}^{G'} W \neq 0$. Show that the same result remains true if we replace $\text{c-Ind}_{H'}^{G'} W$ by any non-0 G-submodule V.

Lemma 16.7. *The image of* $\beta \in H^2(G, \mathbb{C}^\times)$ *under the restriction map* $H^2(G, \mathbb{C}^\times) \to H^2(P, \mathbb{C}^\times)$ *is* 0.

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$$\begin{aligned} (17.1)\quad {}_P\omega_\chi(g)f(w_-^0) &= f(w_-^0 g) \\ &= \tilde\chi\Big(g \ltimes [A_{+-}w_-^0; 0] \cdot \big[0; \tfrac{1}{2}\langle A_{--}w_-^0, A_{+-}w_-^0\rangle\big]\Big) f(A_{--}w_-^0) \\ &= \chi\big(\tfrac{1}{2}\langle A_{--}w_-^0, A_{+-}w_-^0\rangle\big) f(A_{--}w_-^0). \end{aligned}$$

$$\begin{aligned} (17.2)\quad \omega_\chi([w; z])f(w_-^0) &= f(w_-^0[w; z]) \\ &= \tilde\chi\Big([w_+; 0] \cdot \big[0; z + \tfrac{1}{2}\langle w_-, w_+\rangle + \langle w_-^0, w_+\rangle\big]\Big) f(w_-^0 + w_-) \\ &= \chi(z + \tfrac{1}{2}\langle w_-, w_+\rangle)\chi(\langle w_-^0, w_+\rangle) f(w_-^0 + w_-). \end{aligned}$$

Exercise 17.3. Show that

$$\langle A_{++}w_+, A_{--}w_-\rangle = \langle w_+, w_-\rangle$$

and

$$\langle A_{+-}w_-, A_{--}w'_-\rangle = \langle A_{+-}w'_-, A_{--}w_-\rangle$$

for $w_+ \in \mathcal{W}_+$ and $w_-, w'_- \in \mathcal{W}_-$.

Exercise 17.4. Use Exercise 17.3 to verify directly that (17.1) gives an intertwining operator of $g \cdot \omega_\chi$ with ω_χ, and that the resulting map $P \to \mathrm{GL}(V_\chi)$ is a representation of P.

Definition 17.5. Let Λ be a self-dual lattice in $\mathcal{W}$, and $(J_{\Lambda,j})_{j\in\mathbb{R}}$ the associated filtration, as in Definition 9.6. For convenience, we write $\Lambda_\pm$ for $\Lambda \cap \mathcal{W}_\pm$. We may, and do, assume that $\Lambda = \Lambda_+ \oplus \Lambda_-$ (by changing either the lattice Λ or the polarisation $(\mathcal{W}_+, \mathcal{W}_-)$, if necessary). Since the $\mathcal{O}$-module Λ supports a symplectic pairing (the restriction of the pairing on $\mathcal{W}$), it makes sense to consider the group $\mathrm{Sp}_{\mathcal{O}}(\Lambda)$; and, since Λ is a lattice in $\mathcal{W}$, we have a canonical injection $\mathrm{Sp}_{\mathcal{O}}(\Lambda) \hookrightarrow G$. Write K for the image of this injection.

Exercise 17.6. Show that, if $\dim \mathcal{W} = 2$, then $G = PK$.