Mini-workshop in complex dynamics
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Lecture 3

Polynomial dynamics at infinity

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(w. C. Favre, www.arxiv.org)
"Dynamical compactifications of C^2"

Polynomial Mappings

\[ f : \mathbb{C}^2 \to \mathbb{C}^2 \] poly, dominant

Study behavior at \( \infty \) of \( f^n \)

Similar to \( g : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0) \) but

- \( f \) not proper in general so can't say \( f(\infty) = \infty \)

- Situation less local than \((\mathbb{C}^2,0)\) :
  "holomorphic objects" defined near \( \infty \) in \( \mathbb{C}^2 \) extend to all of \( \mathbb{C}^2 \) (Hartogs)

Strategy: combine methods from
- Lecture 1 (global merom selfmaps)
- Lecture 2 (local dynamics)
Dynamical degrees

\( \lambda_2 = \text{top. deg of } f \)
\( \lambda_1 = \lim_{n \to \infty} (\deg f^n)^{1/n} \)

(Assume \( \lambda_1 > 1 \) for the most part).

Thm A: 2 possibilities:
   a) \( \deg f^n \sim \lambda_1^n \)
   b) \( \deg f^n \sim n \cdot \lambda_1^n \). In this case, \( f \) conjugate to a skew product
      \( (x,y) \mapsto (P(x),Q(x,y)) \) (with cond. on \( P, Q \))

Thm B: \( \lambda_1 \) quadratic integer: \( \lambda_1^2 = A \lambda_1 + B \quad A, B \in \mathbb{Z} \)

Thm C: \( \deg f^n \) satisfies recursion formula:
      \( \deg f^n = \sum_{j=1}^{n} a_j \deg f^{n-j} \quad a_j \in \mathbb{Z} \)

Can approach these results by studying dynamics at \( \infty \).
Compactifications

Fix embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$
(=> can talk about affine fans on $\mathbb{C}^2$)

Def: An admissible compactification $X \supseteq \mathbb{C}^2$ is obtained from $\mathbb{P}^2$ by finitely many blowups at $\infty$

Def: Primes of $X = \text{irr. comp's of } X \setminus \mathbb{C}^2$
(e.g. $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$)
Dynamical compactifications

**Thm D:** Assume \( \lambda_2 < \lambda_1 \), "low top deg."

Then there exists:
- An admissible comp'n \( X \supset C^2 \)
- A point \( p \in X \setminus C^2 \)
- An integer \( N \geq 1 \)

**s.t.**
- the lift \( \tilde{\phi} : X \rightarrow X \) is holo at \( p \), \( \tilde{\phi}(p) = p \)
- the germ \( \tilde{\phi} : (X, p) \rightarrow (X, p) \) is superthin
  and admits "simple" normal form
- \( \tilde{\phi}^n E = \{ p \} \) for all primes \( E \) of \( X \)
  (with at most one exception)

Can deduce Thms B, C (when \( \lambda_2 < \lambda_1 \)) from this.

Can also deduce that Green fcn

\[
G := \lim_{n \to \infty} \lambda_1^n \log^+ \| \tilde{\phi}^n \|
\]

is well behaved

Focus on proving Thm A, D
The Riemann-Zariski approach

\[ X := \varinjlim X \quad X \text{ admissible conn' } \]

Riemann-Zariski space at \( \infty \) (don't touch \( \mathbb{C}^2 \))

\[ W(X) := \varinjlim H^{\dim} R(X) \quad \text{Weil classes} \]
\[ C(X) := \varinjlim H^{\dim} R(X) \quad \text{Cartier} \]

\[ L^2(X) \]
\[ \text{Nef}(X) \]

\( \Phi_x, \Phi^x \) act on \( W, C, L^2, \text{Nef} \)

\[ \text{[Use } \Phi \text{ holo on } \mathbb{C}^2 \text{ and } \Phi(\mathbb{C}^2) < \mathbb{C}^2 \text{]} \]

**Thm:** Assume \( \lambda_1 > \lambda_2 \). Then:

a) \( \exists \Theta_x, \Theta^x \in \text{Nef}(X) \), unique up to scaling, s.t. \( \Phi_x \Theta_x = \lambda_1 \Theta_x \), \( \Phi^x \Theta^x = \lambda_2 \Theta^x \)

b) \( \alpha \in L^2(X) \implies \)

\[ \lambda_1^{-n} \Phi^x \alpha \longrightarrow \frac{(\Theta^x \cdot \alpha)}{(\Theta^x \cdot \Theta^x) \Theta_x} \Theta_x \quad (\text{fast}) \]

(\text{+ same for } \Phi^x)

Will interpret \( \Theta_x \)
Valuations

\[ R = \mathbb{C}[x,y] \text{ coordinate ring of } \mathbb{C}^2 \]

\[ \hat{\mathfrak{V}} = \{ \text{valuations } \nu : R \to (-\infty, +\infty] \]

centered at \( \infty \):

\[ \nu(P) < 0 \text{ for some polynomial } P \]

\[ \hat{\mathfrak{V}}_0 = \{ \text{normalized valuations in } \hat{\mathfrak{V}} \}

Normalization:

\[ \min \{ \nu(L) \mid L \text{ affine } \} = -1 \]

\[ \iff \nu(L) = -1 \text{ for generic } L \]

Prime \( E \subset X \) \text{ divisorial valuations}

\[ \text{ord}_E \in \hat{\mathfrak{V}}_0 \]

\[ \nu_E \in \hat{\mathfrak{V}}_0 \]

\[ \nu_E = b_E^{-1} \text{ord}_E \]

\[ b_E = -\text{ord}_E(L) \]

Ex: \( E = L_{\mathbb{A}_2} = \mathbb{P}^2 - \mathbb{C}^2 \Rightarrow \]

\[ \nu_E = \text{ord}_E = -\text{deg} \quad (b_E = 1) \]
Dual graphs

$X$ admissible comp'hn of $\mathbb{C}^2$

$\Gamma_x$ dual graph:
- partial ordering (root = $L_\infty$)
- metric

$\text{dist}(E, F) = \frac{1}{b_p b_F}$

$X' \geq X$ (can get from $X$ to $X'$ by blowing up)

$\Rightarrow \Gamma_x \hookrightarrow \Gamma_{x'}$: order-preserving isometry

Also have:
- Embedding: $\Gamma_x \hookrightarrow Y_0$
- Retraction: $Y_0 \twoheadrightarrow \Gamma_x$

Thm: $Y_0 \cong \varprojlim \Gamma_x$

Cor: $Y_0$ is an $\mathbb{R}$-tree
Valuative dynamics

\[ f : C^2 \rightarrow C^2 \text{ poly, dominant} \]

Want to define:

\[ f_0 : \mathcal{Y}_0 \rightarrow \mathcal{Y}_0 \]

as in local case.

**Problem:** \( f_0 \mathcal{Y} \) may not be centered at \( \mathcal{Y}_0 \).

Get subtree \( D_f \subseteq \mathcal{Y}_0 \) on which \( f_0 \) is defined.

\[ f_0 : D_f \rightarrow \mathcal{Y}_0 \]

respects tree structure, has dense image.

**Problem:** \( D_f \) depends on \( f \).

Is \( D_f \cap D_{f^n} \cap \ldots \cap D_{f^n} \ldots \neq \emptyset \) ?

**Answer:** YES!
The subtree $\mathcal{X}_i : I$

The R-tree $\mathcal{X}_0$ admits two natural parametrizations:

$\alpha : \mathcal{X}_0 \rightarrow [-\infty, 1]$ "skewness"
$\rho : \mathcal{X}_0 \rightarrow [-2, \infty]$ "thinness"

1) $\alpha(v) = 1 - \text{dist}(v, \text{root})$

2) Thinness $\rho$ defined as "log-discrepancy"

$\rho(\text{ord}_E) = 1 + \text{ord}_E(\omega)$ \hspace{1cm} $\omega = dx \wedge dy \wedge dz$

$\rho(v) = b^{-1} \rho(\text{ord}_E)$

Def: $v \in \mathcal{X}_i$ iff \[ \begin{cases} \alpha(v) \geq 0 \\ \rho(v) \leq 0 \end{cases} \]

"$v$ close enough to the root of $\mathcal{X}_0$"
The subtree $\mathcal{Y}_i$ II

Valuations in $\mathcal{Y}_i$ have good properties

Thm: If $\alpha(v) > 0$ (e.g. $v$ in "interior" of $\mathcal{Y}_i$) then
1) $v(p) < 0$ for every nonconstant poly $p$
2) $v(p) \leq -\alpha(v) \cdot \deg p$

Thm: If $\alpha(v) = 0$, $A(v) < 0$ and $v$ is divisorial (⇒ $v$ endpt in $\mathcal{Y}_i$) then $v$ is a rational pencil valuation:

\[ v(q) = \text{const} \cdot \text{ord}_{x=\text{const}} (q|_{p=\text{const}}) \]
Tight compactifications

$X \subset \mathbb{C}^2$ admissible compactification

**Def**: $X$ is tight if $\forall$ prime $E$ of $X$, the valuation $v_E$ lies in $\mathbb{C}^*$

[Restriction on blowups from $\mathbb{P}^2$ to $X$]

Tight compactifications $X \subset \mathbb{C}^2$

have nice geometric properties:

- $\text{Nef}(X)$ is a simplicial cone
- Every nef line bundle on $X$ is generated by global sections
Dynamics on $\mathcal{Y}_t$  I

$f : \mathbb{C}^2 \to \mathbb{C}^2$

Thm: $f$, well defined on $\mathcal{Y}_t$, ($D_f = \mathcal{Y}_t$)
and $f \cdot \mathcal{Y}_t \subset \mathcal{Y}_t$

Cor: Exists eigenvalue $\lambda_\alpha \in \mathcal{Y}_t$

\[ f \cdot \alpha = \lambda \alpha \]

$\gamma \cdot \alpha = \gamma \lambda \alpha$

Proof of Thm A [Behavior of $\deg f^n$].

(i) $\lambda_\alpha$ rational pencil valuation
\[ \Rightarrow f \sim \text{skew product} \]

(ii) $\alpha(\gamma_\alpha) > 0 \Rightarrow \lambda_\alpha^n \leq \deg f^n \leq D \lambda_\alpha^n$
where $D = \alpha(\gamma_\alpha)^{-1}$.

[(iii) one more case ...]

Rem: $\lambda_\alpha < \lambda_1 \Rightarrow \gamma_\alpha$ not divisorial

Rem: $A(\gamma_\alpha) = 0 < \alpha(\gamma_\alpha) \Rightarrow$

$f$ counterexample to $\mathcal{FC}$!
**Dynamics of \( \mathcal{Y} \), II**

**Thm:** Assume \( \lambda_1 > \lambda_2 \). Then:
\[ f^n \nu \to \nu^* \text{ as } n \to \infty \]
for all \( \nu \in \mathcal{Y} \) with at most one exception \( \nu^* \), for which \( f_{\nu} \nu = \nu^* \).

**Pf:** Associate Weil class \( Z_\nu \in W(X) \)
to any valuation \( \nu \in \mathcal{Y}_0 \).
\( \nu \in \mathcal{Y}_1 \Rightarrow Z_\nu \neq f \)
\( f_* Z_\nu = Z_{f_* \nu} \)
\( \Rightarrow \ldots \Rightarrow \)
\( f^n \nu \to \nu^* \) unless
\( (Z_\nu, \Theta^*) = 0 \Rightarrow Z_\nu = \Theta^* \quad \Box \)

**Construction of** \( Z_\nu \) **when** \( \nu = \text{ord}_E \)

\( E \) **prime of** \( X \), \( Z_{\text{ord}_E} = Z_E \) **defined by**:
\[ (Z_E \cdot F) = \begin{cases} 1 & F = E \\ 0 & F \neq E \end{cases} \]

[General \( \nu \) by homog. + approx.]
Proof of Thm D

Thm D: Assume $\lambda_2 < \lambda_1$. Then $\exists X, p, N$ ...

$\Phi$: $(X, p) \supset$ superab. fixed pt genus
$\Phi'' E = p \ \forall E$.

\textbf{Pf:} $\nu_\ast$ cannot be divisorial ($\lambda_2 < \lambda_1$)

- Assume $\nu_\ast$ irrational quasi-monomial
- Successively blow up center of $\nu_\ast$ many times
- Get tight compactification $X$,
  $p = \text{center of } \nu_\ast$ on $X$

- $\nu_\ast$ locally attracting $\Rightarrow$
  $\psi, U(p) \subset U(p) \Rightarrow \Phi$ hole at $p$ etc
- $\psi'' \nu_E \rightarrow \nu_\ast \ \forall E \Rightarrow \Phi'' E = \{p\}$.