Pluripotential theory in a non-archimedean setting

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NUS, Jan 6, 2011
Plan

- Archimedean pluripotential theory.
- Non-archimedean pluripotential theory.
- Joint work with S. Boucksom and C. Favre.
Pluripotential theory on Kähler manifolds

- Pluripotential theory = study of plurisubharmonic (psh) fcns.
- $(X, \omega)$ compact Kähler manifold. Assume $\int_X \omega^n = 1$.
- Say $\varphi : X \to [-\infty, \infty]$ is $\omega$-psh if $dd^c \varphi + \omega \geq 0$.
- Example: $X = \mathbb{P}^n$, $\omega =$ Fubini-Study and

$$\varphi = \frac{1}{2m} \log \sum_{k=1}^N \frac{|f_k|^2}{\| \cdot \|^{2m}},$$

where $f_1, \ldots, f_N$ homo polys on $\mathbb{C}^{n+1}$ with $\bigcap_k f_k^{-1}(0) = \{0\}$.

- **Compactness property**: the function

$$\text{PSH}(X, \omega) \ni \varphi \mapsto \sup_X \varphi \in \mathbb{R}$$

is continuous and proper in the $L^1$-topology.
Monge-Ampère equation: given probability measure $\mu$ on $X$, find $\omega$-psh function $\varphi$ on $X$ such that

$$\text{MA}(\varphi) := (\omega + dd^c \varphi)^n = \mu.$$ 

Thm by Calabi/Yau: uniqueness/existence when $\mu > 0$ smooth.

Thm by Guedj-Zeriahi/Dinew: existence/uniqueness when $\mu$ non-pluripolar measure (no mass on $\{u = -\infty\}$, $u$ psh).

Example: $X = \mathbb{P}^1$, $\omega =$ Fubini-Study.

$$\varphi(z) = \int \log \frac{|z - w|}{\sqrt{1 + |w|^2}} d\mu(w).$$

Non-linear problem in dim $> 1$!
Capacity and extremal functions

▶ To study MA eqn, useful to develop capacity theory.
▶ For any subset \( E \subseteq X \), define extremal function

\[
u_E = \sup \{ \varphi \in \text{PSH}(X, \omega) \mid \varphi \leq 0, \varphi|_E \leq -1 \}.
\]

▶ For \( E \subseteq X \) Borel, define the capacity

\[
\text{Cap}(E) = \sup \left\{ \int_E \text{MA}(\varphi) \mid \varphi \in \text{PSH}(X, \omega), -1 \leq \varphi \leq 0 \right\}.
\]

▶ **Thm**: for \( K \subseteq X \) compact, \( \text{Cap}(K) = \int_K \text{MA}(u_K^*) \) and \( \text{supp} \text{MA}(u_K^*) \subseteq K \).
▶ **Thm**: for any \( E \subseteq X \), TFAE:
  ▶ \( E \) is pluripolar, i.e. \( E \subseteq \{ u = -\infty \} \), \( u \) psh;
  ▶ \( \forall \varepsilon \exists G \supseteq E \) open with \( \text{Cap}(G) < \varepsilon \);
  ▶ \( u_E^* \equiv 0 \);
  ▶ \( E \) is “negligible”. 

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Key tools

- **Regularization**: can approximate any $\varphi \in \text{PSH}(X, \omega)$ by a *decreasing sequence* $\varphi_j \downarrow \varphi$, with $\varphi_j \in \text{PSH}(X, \omega) \cap C^\infty(X)$.

- **Comparison principle**: 
  \[
  \int_{\varphi < \psi} \text{MA}(\psi) \leq \int_{\varphi < \psi} \text{MA}(\varphi)
  \]
  for $\varphi, \psi \in \text{PSH}(X, \omega)$ sufficiently regular.

- **“Balayage”**: Can solve $\text{MA} = 0$ with Dirichlet boundary condition. Implies $\text{supp} \text{MA}(u^*_K) \subset K$.

- **Countability**: $X$ has countable basis for topology. Used to prove e.g. Choquet’s Lemma.
Beyond Archimedes

- Try to extend previous analysis to non-archimedean setting.
- Natural for problems of arithmetic nature.

- **Definition.** Archimedean$\neq$ not non-Archimedean.
- **Factoid:** Archimedes died 2222 years ago.
Assume $k$ field equipped with non-archimedean norm: $|a + b| \leq \max\{|a|, |b|\}$. Also assume $k$ complete.

Example: $C((t))$ (Laurent series) $0 < |t| < 1$, $|c| = 1$.

Example: any field $k$ with trivial norm: $|a| = 1$, $a \neq 0$.

Can try to develop analytic geometry as over $\mathbb{C}$.

Problem 1: $k$ totally disconnected.

Problem 2: in general, $k$ not locally compact.

Various ways to deal with this.

Approach: replace $k$ (or $k^n$) by suitable Berkovich space (space of valuations).
The Berkovich affine space

- **Def**: $A^n_{Berk}$ is the set of multiplicative seminorms on $k[z_1, \ldots, z_n]$ extending the given norm on $k$.
- When $k = \mathbb{C}$, $A^n_{Berk} = \mathbb{C}^n$ by Gelfand-Mazur.
- Points of $A^1_{Berk}$ can be viewed as *balls* in $k$. Thus $A^1_{Berk}$ admits *tree structure*.

- For $n > 1$, $A^n_{Berk}$ harder to visualize!
Try to adapt previous results to Berkovich spaces. Look at special case: \( k = \mathbb{C} \) with trivial valuation.

Define subsets \( \mathcal{V} \subset \hat{\mathcal{V}} \subset A^n_{\text{Berk}} \) by

\[
\hat{\mathcal{V}} := \{ \text{seminorms on } \mathbb{C}[z_1, \ldots, z_n] \mid \max_i |z_i| < 1 \}.
\]

\[
\mathcal{V} := \{ \text{seminorms on } \mathbb{C}[z_1, \ldots, z_n] \mid \max_i |z_i| = e^{-1} \}.
\]

Useful for studying singularities at a point in \( \mathbb{C}^n \).

\( \hat{\mathcal{V}} \cong \) cone over \( \mathcal{V} \).

\( \mathcal{V} \) contractible compact Hausdorff space. No countable basis.

\( \mathcal{V} \cong \) limit of simplicial complexes. \( \mathbb{R}\)-tree when \( n = 2 \).

Define class \( \text{PSH}(\mathcal{V}) \) and “do” pluripotential theory on it.

Functions in \( \text{PSH}(\mathcal{V}) \) extend to \( \hat{\mathcal{V}} \) by homogeneity.
▶ View elements of $\mathcal{V}$ as *semivaluations*

$$\nu : \mathbb{C}[z_1, \ldots, z_n] \rightarrow [0, +\infty]$$

using $| \cdot | = e^{-\nu}$.

▶ Define psh fcn as *decreasing limit* of fcns of the form

$$c \max_{1 \leq j \leq N} \log |f_j|,$$

where $c > 0$, $f_j \in \mathbb{C}[z_1, \ldots, z_n]$ and $\bigcap_j \{f_j = 0\} = \{0\}$.

▶ Here $\log |f|_{(\nu)} := -\nu(f)$.

▶ Can define analogy of $L^1$-topology (but not metrizable).

▶ **Thm**: the function

$$\text{PSH}(\mathcal{V}) \ni \varphi \mapsto \sup_\mathcal{V} \varphi \in \mathbb{R}_-$$

is continuous and proper.

▶ Proof uses multiplier ideals to construct $\varphi_j \searrow \varphi$. 
Monge-Ampère operator on $\mathcal{V}$

- First define $\text{MA}(\varphi)$ for $\varphi = \max_j \log |f_j|$.
- Suffices to define $\int_{\mathcal{V}} \psi \text{MA}(\varphi)$ for $\psi = \max_i \log |g_i|$.
- Do this as a local intersection number. Analytically:
  \[ \int_{\mathcal{V}} \psi \text{MA}(\varphi) = -((dd^c \varphi)^{n-1} \wedge dd^c \psi)\{0\}, \]
  where in the RHS we view $\varphi$ and $\psi$ as functions on $\mathbb{C}^n$!
- For general $\varphi \in \text{PSH}(\mathcal{V})$, define $\text{MA}(\varphi) = \lim_j \text{MA}(\varphi_j)$, where $\varphi_j \searrow \varphi$.
- **Thm.** Unique solution to $\text{MA}(\varphi) = \mu$ for any non-pluripolar measure $\mu$.
- Existence proof uses variational approach (Alexandrov; Berman-Boucksom-Guedj-Zeriahi).
- Need to develop capacity theory along the way.
For any subset $E \subseteq \mathcal{V}$, define extremal function

$$ u_E = \sup\{\varphi \in PSH(\mathcal{V}) \mid \varphi|_{E} \leq -1\}. $$

For $E \subseteq \mathcal{V}$ Borel, define the capacity

$$ \text{Cap}(E) = \sup\{ \int_{E} MA(\varphi) \mid \varphi \in PSH(\mathcal{V}), \varphi \geq -1\}. $$

**Thm:** for $K \subseteq \mathcal{V}$ compact, $\text{Cap}(K) = \int_{K} MA(u_{K}^{*})$ and $\text{supp} MA(u_{K}^{*}) \subseteq K$.

**Thm:** for any $E \subseteq \mathcal{V}$, TFAE:

1. $E$ is pluripolar, i.e. $E \subseteq \{u = -\infty\}$, $u$ psh;
2. $\forall \epsilon \exists G \supseteq E$ open with $\text{Cap}(G) < \epsilon$;
3. $u_{E}^{*} \equiv 0$;
4. $E$ is “negligible”.

Results exactly parallel to the archimedean situation!
Many proofs are the same as in the archimedean situation, but a few basic ingredients are different.

**Countability**: $\mathcal{V}$ has no countable basis for topology but $\mathbb{C}[z_1, \ldots, z_n]$ is noetherian. Used to prove Choquet’s Lemma.

**Regularization**: can approximate any $\varphi \in \text{PSH}(\mathcal{V})$ by a decreasing sequence $\varphi_j \downarrow \varphi$, with $\varphi_j \in \text{PSH}(\mathcal{V})$ of nice form.

This was built into the definition of $\text{PSH}(\mathcal{V})$... but then properness of $\varphi \rightarrow \sup \varphi$ was hard to establish!

In $\mathbb{C}^n$, can regularize using convolution. On $\mathcal{V}$, we use multiplier ideals.
Key tools II

- **Comparison principle:**

\[
\int_{\varphi < \psi} \text{MA}(\psi) \leq \int_{\varphi < \psi} \text{MA}(\varphi)
\]

for \(\varphi, \psi \in \text{PSH}(\mathcal{V})\) not too singular.

- As in \(\mathbb{C}^n\) prove this by reduction to the case \(\varphi, \psi\) “smooth”.

- In \(\mathbb{C}^n\), the smooth case uses Stokes’ theorem. On \(\mathcal{V}\), uses positivity of certain intersection nos.

- **“Balayage”:** Don’t know how to solve Dirichlet problem locally. Replacement is the *orthogonality property* of asymptotic Zariski decompositions in the sense of Boucksom-Demailly-Păun-Peternell.
Future work

- Would like to do pluripotential theory on other Berkovich spaces.
- Example: work on all of $\mathbb{A}_\text{Berk}^n$, not just localized at one point.
- Example: analytification of projective varieties over $\mathbb{C}((t))$. 