

Algebraic singularities in two dimensions

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1. INTRODUCTION

These notes aim at giving a brief introduction to some of my own research on singularities in algebraic geometry. They do *not* try to give a general overview of the subject and certainly do not give credit to all researchers in the field. Use them at your own risk!

2. SINGULARITY THEORY

Roughly speaking, singularity theory in algebraic geometry is the study of the local structure of solutions to polynomial equations over the complex numbers:

$$P_1(x_1, \dots, x_n) = \dots = P_m(x_1, \dots, x_n) = 0. \quad (2.1)$$

The solution set is thus a subset of \mathbf{C}^n and we want to understand its structure near one of its points. In algebraic geometry, one often wants to work over other fields (or rings) than the complex numbers \mathbf{C} , but I will do so here.

Sometimes the subset of \mathbf{C}^n defined by (2.1) looks locally like a complex manifold (of some dimension $< n$); otherwise we say that it has a *singularity*. Two examples of singularities are given by the *cuspidal curve* $y^2 = x^3$ in \mathbf{C}^2 and the *Whitney umbrella* $x^2 = y^2z$.

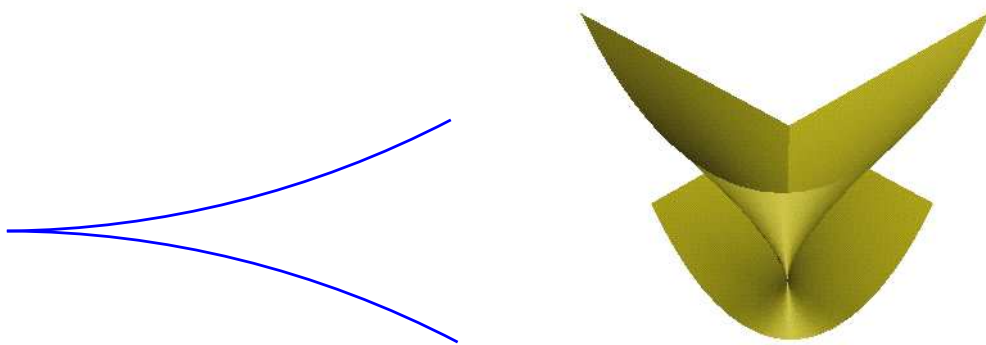


FIGURE 1. The cusp (left) and the Whitney umbrella (right)

I will focus on the case of singularities in two dimensions, at the origin in \mathbf{C}^2 . This is much simpler than the general situation, but some aspects are surprisingly subtle. The most interesting part of my approach (joint with C. Favre) is that it can be used to analyze singularities of more general objects, such as plurisubharmonic functions. At least part of the techniques should be possible to use in higher dimensions, too, albeit with substantial additional difficulties.

Quite concretely, then, we want to understand the solution set—near the origin in \mathbf{C}^2 —to finitely many polynomial equations in two variables:

$$\phi_1(x, y) = \dots, \phi_m(x, y) = 0.$$

In fact, since the point of view is rather analytic, we may as well assume that the ϕ_i are just germs of holomorphic functions at the origin in \mathbf{C}^2 . To avoid trivialities we should assume that $\phi_i(0, 0) = 0$ for all i .

If $m = 1$, then the solution set is a curve which may or may not be singular.

If $m = 2$, then typically the solution set is just the origin, but for many purposes it is useful to retain the information of the defining equations in terms of the *ideal* $I = \langle \phi_1, \dots, \phi_m \rangle$ generated by the ϕ_i . The ideal lives inside, say, the ring \mathcal{O}_0 of holomorphic germs near the origin in \mathbf{C}^2 .

3. RESOLUTION OF SINGULARITIES

An important idea in singularity theory is that singularities can often be *resolved*, or simplified. Formulating this precisely involves some technicalities. Let us illustrate the idea on the cusp $y^2 = x^3$.

A first attempt may be to view the cusp as the shadow of a three-dimensional smooth curve. This is indeed possible. If C is the curve in \mathbf{C}^3 with parameterization $t \mapsto (t^2, t^3, t)$ then C is smooth and maps onto the cusp under the projection $(x, y, z) \rightarrow (x, y)$. A problem with this method is that the dimension of the underlying space has increased, something that often is desirable to avoid.

A second attempt is to make a change of variable in order to render the cusp smooth. This may seem silly, since the basic shape of a curve does not depend on the coordinates chosen. However, if we allow somewhat wild changes of variables, then we can achieve something. In the case of the cusp, consider the changes of variables

$$(z, w) \mapsto (x, y) = (z^2(1 + w), z^3(1 + w^2)^2).$$

It is easy to see that the smooth curve $w = 0$ maps to the cusp $y^2 = x^3$. Moreover, we can solve for (z, w) in terms of (x, y) so the coordinate change is essentially one-to-one. However, the curve $z = 0$ is collapsed to the origin $(x, y) = (0, 0)$. What we have is a *birational* change of coordinates and this is good enough for many purposes. So we have resolved the singularity of the cusp. The only problem is that the coordinates change we made was an intelligent guess: can we resolve curve singularities more systematically?

The answer to this last question is yes, and involves the process of *blowups*. Before explaining this, it is convenient, and even necessary to modify the goal of the resolution process. Rather than looking for smooth curves, we shall look for curves with *normal crossing singularities*. In dimension two this means a curve that at each point is either smooth or looks like the curve $\{xy = 0\}$, that is, two smooth curves intersecting transversely.

Concretely, but also somewhat naively, the process of blowing up a point in \mathbf{C}^2 means performing the birational change of coordinates $(x, y) \mapsto (x, xy)$. The preimages of the lines $\{y = \lambda x\}$ through the origin contain the *disjoint* lines $\{y = \lambda\}$ but they also all contain the line $\{x = 0\}$, which is the preimage of the origin. As a piece of terminology, $\{y = \lambda\}$ is the *strict transform* of $\{y = \lambda x\}$ whereas $\{y = \lambda\} \cup \{x = 0\}$ is the *total transform*.

Notice that in these coordinates, the strict transform of $\{x = 0\}$ is empty; it is located “at infinity”. More geometrically (and canonically), one can define a complex surface X and a map $\pi : X \rightarrow (\mathbf{C}^2, 0)$. The latter is holomorphic and one-to-one off the *exceptional divisor* $\pi^{-1}(0) \simeq \mathbf{P}^1$. This construction replaces the origin in \mathbf{C}^2 by a rational curve (Riemann sphere) inside X .

We can iterate the process of blowing up. In the case of the cusp $C := \{y^2 = x^3\}$ one can perform three consecutive blowups and obtain $\pi : X \rightarrow (\mathbf{C}^2, 0)$ such that the total transform $\pi^{-1}(C)$ has normal crossings singularities.

In fact as the following result (due to M. Noether) shows, singularities of plane curves can always be turned into normal crossings after finitely many blowups.

Theorem 3.1. *For any curve $C \subset (\mathbf{C}^2, 0)$ there exists a finite composition of point blowups $\pi : X \rightarrow (\mathbf{C}^2, 0)$ such that the total transform $\pi^{-1}(C)$ has normal crossings singularities.*

The strategy of the proof is simple: perform a blowup whenever you see a singularity that is not a normal crossing. The nontrivial part of the proof consists of showing that this procedure terminates in finite time.

It is a highly nontrivial result by Hironaka that resolution of singularities—suitably interpreted—is possible in higher dimensions, too.

4. SINGULARITY EXPONENTS AND THE ASCENDING CHAIN CONDITION

Resolution of singularities is of fundamental importance for many applications, but in itself does not give an idea of how complicated a singularity is.

For curves in $(\mathbf{C}^2, 0)$, a first measurement of a singularity is given by the multiplicity. If $C = \{\phi = 0\}$ is a curve, defined by a holomorphic germ ϕ , then the *multiplicity* $m(C)$ of C is defined as the largest integer m such that $|\phi(p)| \lesssim \|p\|^m$ as $p \rightarrow 0$; this does not depend on the choice of ϕ .

It is easy to see that $m(C) = 1$ iff C is smooth so the multiplicity gives some idea of the complexity of a singularity. However, the curves $\{y^2 = x^{2n+1}\}$, $n \geq 1$ all have multiplicity two, but arguably the complexity increases as n increases.

A different way of measuring the singularity is in terms of integrability. Pick a holomorphic germ ϕ at the origin in \mathbf{C}^2 with $\phi(0) = 0$. It is easy to see that the function $|\phi|^{-1}$ fails to be integrable at the origin. However, $|\phi|^{-c}$ will be integrable for small $c > 0$ and we can define the *singularity exponent* or *log-canonical threshold* of ϕ as $c(\phi)$, where

$$c(\phi) := \sup\{c > 0 \mid |\phi|^{-2c} \in L_{\text{loc}}^1\}.$$

For our purposes, the reciprocal $\lambda(\phi)$ of $c(\phi)$, known as the *Arnold multiplicity* of ϕ , will be more natural. Trivially:

$$\lambda(\phi) := \inf\{\lambda > 0 \mid |\phi|^{-2/\lambda} \in L_{\text{loc}}^1\}.$$

Notice that $\lambda(\phi^k) = k\lambda(\phi)$ for any holomorphic germ ϕ and any $k \geq 1$. As for the germs discussed above, it is easy to see that

$$\lambda(y^2 - x^n) = \frac{2n}{n+2},$$

which is an increasing function of n .

A fundamental property of the singularity exponent is that it satisfies the *Ascending Chain Condition* (ACC). This means that if $(\phi_k)_1^\infty$ is a sequence of germs such that the sequence $(c(\phi_k))_1^\infty$ is increasing, then the latter sequence is stationary, i.e. $c(\phi_{k+1}) = c(\phi_k)$ for large k .

The ACC in dimension two has been known for some time; C. Favre and I recently gave a new proof.

The definition of $c(\phi)$ and $\lambda(\phi)$ makes sense for germs in any dimension and it is conjectured that the ACC holds for the set of singularity exponents $c(\phi)$. This was later proved by McKernan and Prokhorov, conditioned on Mori's minimal model program (MMP). Since MMP is worked out in dimension three, it remains to settle the ACC conjecture in dimensions four and higher.

Usually in algebraic geometry, singularity exponents are defined combinatorially, using a resolution of singularities, but the latter definition is equivalent to the one given here.

5. MULTIPLIER IDEALS

A modification of the singularity exponent yields a finer invariant of a singularity. Consider for instance an ideal I in the ring \mathcal{O}_0 of holomorphic germs at the origin in \mathbf{C}^2 and pick generators ϕ_1, \dots, ϕ_k for I . The *multiplier ideal* $\mathcal{J}(I)$

of I consists of all holomorphic germs $\psi \in \mathcal{O}_0$ such that the function

$$\frac{|\psi|^2}{\sum_1^k |\phi_j|^2}$$

is locally integrable. One can check that $\mathcal{J}(I)$ does not depend on the choice of generators ϕ_j of I . An ideal is determined by its multiplier ideals in the following sense: if I_1 and I_2 are two (integrally closed) ideals, then $I_1 = I_2$ iff $\mathcal{J}(I_1^k) = \mathcal{J}(I_2^k)$ for all $k \geq 1$.

From an analytic point of view it is natural to define multiplier ideals also for (formal) objects of the type I^c , where I is an ideal and $c > 0$. We simply set

$$\mathcal{J}(I^c) := \{\psi \in \mathcal{O}_0 \mid |\psi|^2 / (\sum_1^k |\phi_j|^2)^c \in L_{\text{loc}}^1\}.$$

In dimension two it was proved independently by Lipman-Watanabe and by C. Favre and myself that essentially any ideal in \mathcal{O}_0 can be realized as a multiplier ideal. More precisely, any integrally closed ideal J in \mathcal{O}_0 is of the form $J = \mathcal{J}(I^c)$,

6. THE VALUATIVE TREE

The key to my approach (joint with C. Favre) to singularities is an object called *the valuative tree*. It provides an efficient means of encoding singularities of various objects (locally, in two dimensions). So what is this object?

The precise definition of the valuative tree \mathcal{V} is as the set of $[0, \infty]$ -valued *valuations* on the ring \mathcal{O}_0 , centered at the maximal ideal and suitably normalized.

However, it is more instructive to explain the structure of \mathcal{V} . As the name indicates, it is a kind of tree in the sense of a (large) collection of line segments glued together in such a way that no cycles appear. Figure 2 gives a rough idea of its structure, which is self-similar in a certain sense. However, while the valence at each branch point is four in the picture, it should really be uncountable. Moreover, the set of branch points form a dense subset along any segment.

The segments in \mathcal{V} arise as follows. First, there is a special valuation in \mathcal{V} , the multiplicity valuation $\nu_{\mathfrak{m}}$, given by the order of vanishing at the origin. This valuation $\nu_{\mathfrak{m}}$ serves as the root of \mathcal{V} . Now, to any irreducible curve $C \subset (\mathbf{C}^2, 0)$ we associate a segment in \mathcal{V} joining C and $\nu_{\mathfrak{m}}$. Denote it by $[\nu_{\mathfrak{m}}, C]$. Two such segments $[\nu_{\mathfrak{m}}, C]$, $[\nu_{\mathfrak{m}}, D]$ intersect in a subsegment, the size of which depends on the order of tangency of the curves C and D . One extreme case is $C = D$ in which case the two segments (of course) coincide. The other extreme case is when the curves C and D have distinct tangents at the origin; then the segments meet only at $\nu_{\mathfrak{m}}$. See Figure 3.

If the curve C is smooth, then we can interpret the points on the segment $[\nu_{\mathfrak{m}}, C]$ as follows. Pick local coordinates (x, y) such that $C = \{y = 0\}$. Given

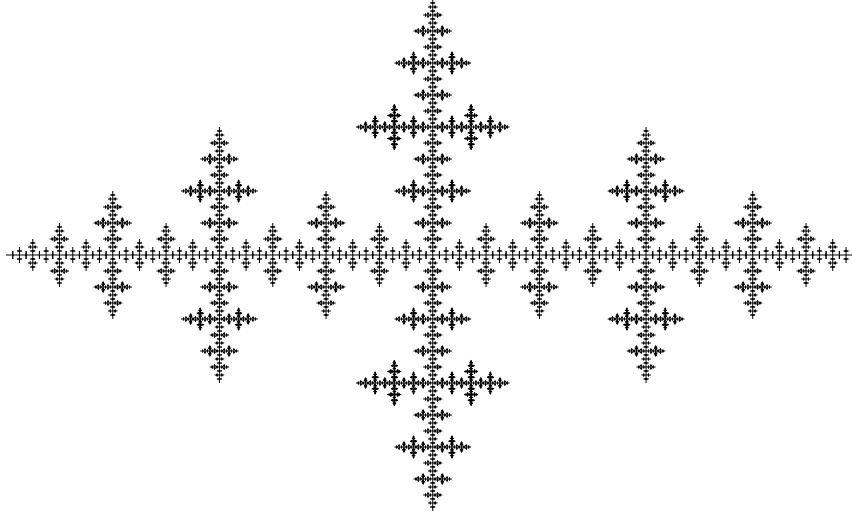


FIGURE 2. The valuative tree. This rough picture emphasizes the self-similar structure but does not explain how the segments arise. See Figure 3 for the latter.

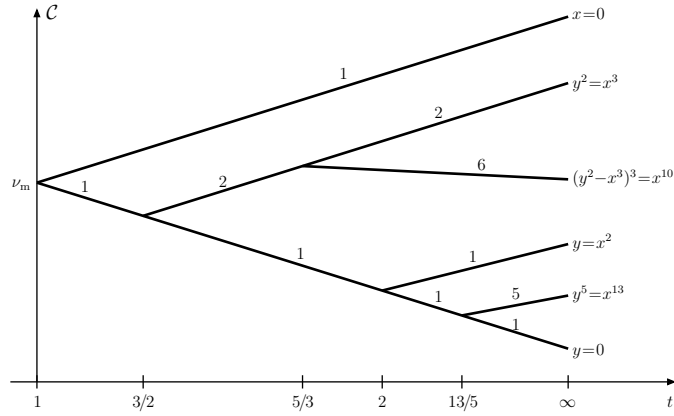


FIGURE 3. The valuative tree. This fails to capture the complexity and self-similarity of the tree but explains how the segments appear.

$t \in [1, \infty]$ define a function $\nu_{C,t}$ on the ring \mathcal{O}_0 of holomorphic germs with values

in $[0, \infty]$ as follows: $\nu_{C,t}(x) = 1$, $\nu_{C,t}(y) = t$, and more generally

$$\nu_{C,t}\left(\sum a_{ij}x^i y^j\right) = \min\{i + jt \mid a_{ij} \neq 0\}.$$

This is a *monomial valuation* in coordinates (x, y) .

If C is singular, the above definition has to be modified slightly, but we still end up with functions $\nu_{C,t} : \mathcal{O}_0 \rightarrow [0, \infty]$ for $1 \leq t \leq \infty$.

7. THE ZARISKI FACTORIZATION OF INTEGRALLY CLOSED IDEALS

As an example how the valuative tree can be used to study singularities, we can give an interpretation of Zariski's celebrated factorization of integrally closed ideals.

First, to any holomorphic germ $\phi \in \mathcal{O}_0$ we associate a function on the valuative tree \mathcal{V} . Recall that the elements of \mathcal{V} are functions on \mathcal{O}_0 . By fixing ϕ we therefore obtain a function g_ϕ on \mathcal{V} , defined by

$$g_\phi(\nu) := \nu(\phi), \quad \nu \in \mathcal{V}.$$

This function g_ϕ is called the *tree transform* of ϕ .

A valuation $\nu \in \mathcal{V}$ can also be evaluated on an ideal $I \subset \mathcal{O}_0$ by declaring

$$\nu(I) := \min\{\nu(\phi) \mid \phi \in I\}.$$

As before, this gives rise to a function g_I on \mathcal{V} , the tree transform of I .

Functions of the form g_ϕ or g_I should be thought of as very special: they behave like piecewise affine functions on the real line. The latter functions can essentially be determined by their Laplacians, which are finite linear combinations of Dirac masses.

In our situation, the same is true. There exists a natural *tree Laplacian* Δ which associates to any tree transform g_ϕ or g_I an atomic *tree measure* ρ_ϕ or ρ_I . It follows from the definitions that if I and J are two ideals, then $\rho_{IJ} = \rho_I + \rho_J$.

Zariski's factorization theorem says that any (integrally closed) ideal $I \subset \mathcal{O}_0$ admits a unique factorization into "simple" (integrally closed) ideal. In terms of tree measures, this factorization corresponds exactly to decomposing the atomic measure ρ_I into atoms.

The valuative tree also played a key role in the proof by Favre and myself on the realization of ideals as multiplier ideals.