Some complex dynamics

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1. Introduction

These notes aim at giving a brief introduction to some of my own research in complex dynamics. They do *not* try to give a general overview of the subject and certainly do not give credit to all researchers in the field. Use them at your own risk!

2. Dynamical systems

A *dynamical system* is a model of a system that undergoes a time evolution. In what follows we will assume that time is *discrete*, i.e. only takes the values $0, 1, 2, \dots$ (in suitable units).[1] The model then consists of:

- a *state space* X;
- a mapping $F : X \to X$.

The state space X, which mathematically is just a set, is supposed to represent all possible states of the model and F represents the rule deciding how the state changes from time n to time $n+1$.

Sometimes it is natural to work with *continuous* time rather than discrete time. This leads to ordinary differential equations; we shall not discuss these here.

Implicitly we are assuming that the system is *deterministic*, i.e. given the state $x \in X$ at time n, we are 100% sure that the state will be $F(x)$ at time $n+1$.[2] We are also assuming that the system is *autonomous*, that is, the state at time $n+1$, depends only on the state at time n, not on n itself.[3]

It is suggestive to write $F : X \circlearrowleft$ rather that $F : X \to X$ to emphasize that we are dealing with selfmaps. The evolution from time k to $k+n$ is determined by the *iterate* $F^n : X \circlearrowleft$ defined by

$$F^n = F \circ \cdots \circ F \quad (n \text{ times}).$$

[1] Sometimes it is natural to allow time to take negative values too, i.e. $n \in \mathbf{Z}$.

[2] Nevertheless, stochastic considerations play an important role even when the system is deterministic.

[3] A *non-autonomous* system would be modeled by a sequence of mappings $F_n : X \to X$.

Often one is interested in *asymptotic* or *long-term* properties, i.e. the behavior of F^n as $n \to +\infty$.

By fixing an initial state and letting time increase we obtain an *orbit*

$$x, F(x), F^2(x), F^3(x), \ldots$$

The simplest orbits are those of *fixed points*, i.e. points $x \in X$ such that $F(x) = x$. Almost as simple are *periodic points*, i.e. points $x \in X$ with $F^n(x) = x$ for some $n > 0$.

An important example of dynamical systems arise in iterative algorithms, such as Newton's method. In this case X is typically a subset of $\mathbf{R}^n$ (or $\mathbf{C}^n$) and the algorithm tries to locate a particular point $x_0 \in X$ (the solution to some equation). The mapping F describes the rule that takes a approximation $x \approx x_0$ of the solution to a new—hopefully more accurate—approximation $F(x)$ of x_0. Then x_0 is a fixed point of F that should be attracting in the sense that the orbit $(F^n(x))_0^\infty$ converges to x_0 as long as the initial point x is sufficiently close to x_0.

3. Categories of dynamics

The definitions so far are too general. For practical purposes one needs to assume that the state space X has additional structure and that $F : X \circlearrowleft$ preserves this structure. This leads to several classes of dynamical systems:

- **Topological dynamics**: X is a topological space and F is a continuous selfmap;
- **Measurable dynamics**: X is a set equipped with a σ-algebra $\mathcal{F}$ and a probability measure $\mu : \mathcal{F} \to [0,1]$; $F : X \circlearrowleft$ is $\mathcal{F}$-measurable; and F preserves the measure μ in the sense that $F_*\mu = \mu$, that is, $\mu(F^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$;
- **Differentiable dynamics**: X is a differentiable manifold and F is smooth;
- **Complex dynamics**: X is a complex manifold and F is holomorphic (or meromorphic).

I shall focus on complex dynamics here, although ideas from e.g. measurable dynamics play an important role even when studying holomorphic mappings.

A word of warning: the word "complex dynamics" refers to "complex variables". In other branches of mathematics, "complex dynamics" rather refers to "complicated dynamics".

4. Fixed points; local and global dynamics

The easiest orbits to understand are those of *fixed points*, that is, points $x \in X$ with $F(x) = x$. One may then try to analyze the behavior of F and its iterates in a small neighborhood of x. We then write $F : (X, x) \circlearrowleft$ for this dynamical system, referred to as the *local dynamics* of F at x. The notation is somewhat sloppy but convenient as there may not exist small neighborhoods U of x that are invariant, that is, $F(U) \subset U$.

From now on we shall consider complex dynamics, i.e. X is a complex manifold and F is holomorphic. If x is a fixed point, we may choose local coordinates at x and identify a small neighborhood of x with a small neighborhood of the origin in $\mathbf{C}^n$.

Local dynamics stands in contrast to *global dynamics*, which concerns the dynamics of F on the whole manifold X, which is often assumed compact. However, the distinction between local and global dynamics is not as large as it may seem. For instance every rational selfmap $F : \mathbf{P}^1 \circlearrowleft$ of the Riemann sphere $\mathbf{P}^1 = \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ lifts to a homogeneous polynomial selfmap $G : \mathbf{C}^2 \circlearrowleft$ and the local dynamics of G at the origin (a fixed point) "contains" all the global dynamics of F. Notice, however, that we have increased the dimension of the state space. Typically, the difficulty of studying a dynamical system is a very rapidly increasing function of the dimension!

5. Fixed points in dimension one

The easiest case of local dynamics is that in dimension one. Then we are studying a selfmap $f : (\mathbf{C}, 0) \circlearrowleft$. Thus f is an analytic function in a neighborhood of the origin in $\mathbf{C}$ with $f(0) = 0$. We can write

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,$$

where the power series converges near the origin.

Two natural questions are:

(i) What is the behavior of typical orbits starting near the origin?
(ii) Can we make a holomorphic change of coordinates such that f takes a particularly nice form?

If $a_1 \neq 0$, then $f(z) \approx a_1 z$ for $z \approx 0$, so the behavior of the local dynamics is essentially determined by the value of a_1. In particular, if $|a_1| < 1$, then we would expect the origin to be an *attracting* fixed point i.e. that the orbits starting near the origin would converge to the origin. If $|a_1| > 1$, on the other hand, then the origin should be a *repelling* fixed point. These expectations can in fact be turned into theorems, and we can answer the question (ii) as follows:

Proposition 5.1. *If $|a_1| \notin \{0, 1\}$, then there exists a local coordinate ζ at the origin such that $f(\zeta) = a_1 \zeta$ in these coordinates.*

The expression $f(\zeta) = a_1\zeta$ is an example of a *normal form*. One advantage of this normal form is that the dynamics becomes completely transparent: for any $n \geq 1$ we have $f^n(\zeta) = a_1^n\zeta$.

The *neutral* case when $|a_1| = 1$ is both subtle and interesting, but will not be discussed here. What is most interesting to me is the *superattracting* case when $a_1 = 0$. Then the origin is indeed attracting and we can find a very simple normal form. Write

$$f(z) = \sum_{k \geq c} a_k z^k,$$

where $c \geq 2$ and $a_c \neq 0$. The number $c = c(f)$ does not depend on the choice of coordinates z.

Proposition 5.2. *If $c(f) = c \geq 2$, then there exists a local coordinate ζ at the origin, in which $f(\zeta) = \zeta^c$.*

The coordinate ζ is called the *Böttcher coordinate* and renders the dynamics transparent: $f^n(\zeta) = \zeta^{c^n}$ for all $n \geq 1$.

An important case of superattracting fixed points arises for *polynomial* maps $F : \mathbf{C} \circlearrowleft$. Then F extends to a holomorphic selfmap $F : \mathbf{P}^1 \circlearrowleft$ of the Riemann sphere $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ and ∞ is a superattracting fixed point for F. If we write f for the restriction of F to a neighborhood of infinity, then $c(f)$ equals the degree of the polynomial F.

6. Superattracting fixed points in two dimensions

Local (complex) dynamics in two dimensions means the study of fixed point germs $f : (\mathbf{C}^2, 0) \circlearrowleft$. Concretely, given local coordinates (x, y) at the origin, f takes the form

$$f(x, y) = (\phi(x, y), \psi(x, y)),$$

where ϕ and ψ are holomorphic function near the origin in $\mathbf{C}^2$ and $\phi(0, 0) = \psi(0, 0) = 0$. Just as in one dimension, the first thing to do is to study the derivative $Df(0)$ of f at the origin. The behavior of f will be influenced by the eigenvalues of $Df(0)$.

For me, the most important case is that of *superattracting* fixed point germs, which means that the differential map $Df(0)$ vanishes identically. It was observed by Hubbard and Papadopol that there cannot be a simple normal form as the Böttcher coordinate. Probably it is very hard to find local normal forms for f in general.

Instead of aiming for normal forms, one can try to study how fast orbits converge to the origin. This is of obvious importance in the case when f represents an iterative algorithm (such as Newton's method). To make things precise, define

$c(f)$ to be the largest integer c such that

$$\|f(p)\| \leq C\|p\|^c, \quad \text{for some } C > 0$$

as $p \to 0$. Thus $c(f) \geq 1$ and $c(f) \geq 2$ iff $Df(0) \equiv 0$. It is not hard to verify that

$$c(f^{n+m}) \geq c(f^n)c(f^m)$$

for all $n, m \geq 1$. It is then an exercise (albeit not completely trivial) to show that the limit

$$c_\infty(f) = \lim_{n\to\infty} c(f^n)^{1/n}$$

exists. It measures the *asymptotic rate of attraction* to the origin. In the case when f represents an iterative algorithm, $c_\infty(f)$ represents its convergence speed.

Example 6.1. If $f(x,y) = (x^c, y^c)$, then $f^n(x,y) = (x^{c^n}, y^{c^n})$ for all $n \geq 1$. Thus $c(f^n) = c^n$ so $c_\infty(f) = c$.

Example 6.2. If $f(x,y) = (y, xy)$ then one can check that the sequence $c(f^n)$ starts with

$$2, 3, 5, 8, 13, 21, 34, \dots,$$

that is, the Fibonacci sequence. This implies that $c_\infty(f) = (\sqrt{5}+1)/2$, the golden mean.

Example 6.3. If $f(x,y) = ((y^2 - x^3)^2, (y^2 - x^3)^3 + x^1 5)$, then one can easily check by explicit computation that $c(f) = 4$ and $c(f^2) = 34 > 4^2$. It is less trivial to verify that $c_\infty(f) = 3 + \sqrt{39} \approx 9.245$.

As the following result by C. Favre and myself shows, these examples illustrate the general behavior:

Theorem 6.4. *For any fixed point germ $f : (\mathbf{C}^2, 0) \circlearrowleft$, the asymptotic rate of attraction $c_\infty = c_\infty(f)$ is a quadratic integer: there exist integers A, B such that*

$$c_\infty^2 = Ac_\infty + B.$$

7. Polynomial maps in two dimensions

Instead of studying local dynamics near the origin in $\mathbf{C}^2$ we could study polynomial maps of $\mathbf{C}^2$ near infinity. Thus we consider a polynomial selfmap $F : \mathbf{C}^2 \circlearrowleft$; concretely this means that

$$F(X,Y) = (P(X,Y), Q(X,Y)),$$

where P and Q are polynomials in (X,Y). We define the *degree* $\deg(F)$ of F to be the maximum of the degrees of P and Q. For a "typical" point p near infinity we have

$$\|F(p)\| \gtrsim \|p\|^{\deg F}$$

so the sequence $(\deg F^n)_1^\infty$ should measure how fast typical orbits approach infinity. The situation is thus quite similar to the one near the origin in $\mathbf{C}^2$. Indeed, it is not so hard to verify that the sequence $(\deg F^n)_1^\infty$ is *submultiplicative*, that is, $\deg F^{n+m} \le \deg F^n \cdot \deg F^m$ for all m, n. This implies that the limit

$$d_\infty(F) := \lim_{n\to\infty} (\deg F^n)^{1/n}$$

exists. We call d_∞ the *asymptotic degree* or the *asymptotic attraction rate* of F. It is also known as the *first dynamical degree* of F.

However, there is one important difference between polynomial maps and superattracting fixed points: even if the degree of F is quite large, F may not be proper, i.e. there could be points near infinity that are mapped to, say, the origin. A simple example is given by the map $F(X,Y) = (Y, X^kY)$. Here the points on the line $Y = 0$ are all sent to the origin.

The possible failure of F being proper leads to technical complications; nevertheless the following analogue of Theorem 6.4 holds:

Theorem 7.1. *For any polynomial map* $F : \mathbf{C}^2 \circlearrowleft$*, the asymptotic degree* $d_\infty = d_\infty(F)$ *is a quadratic integer: there exist integers* A*,* B *such that*

$$d_\infty^2 = Ad_\infty + B.$$

8. Dynamic desingularization

Our main technique for controling the asymptotic rate of attraction (either near the origin or near infinity in $\mathbf{C}^2$) is to prove a sort of dynamic desingularization result.

Recall that desingularization of curves aims at reducing the singularities to normal crossings. Now define a fixed point germ $f : (\mathbf{C}^2, 0) \circlearrowleft$ to be *rigid* if the critical set of f (i.e. the locus where f is not locally invertible) is contained in a totally invariant set C with simple normal crossings. This is equivalent to saying that there exists a curve C with simple normal crossings such that the critical set of f^n is contained in C for all $n \ge 1$.

Theorem 8.1. *For any (dominant) fixed point germ* $f : (\mathbf{C}^2, 0) \circlearrowleft$ *there exists a composition of blowups* $\pi : X \to (\mathbf{C}^2, 0)$ *and a point* p *on the exceptional divisor* $\pi^{-1}(0)$ *such that the lift* $\hat{f}$ *of* f *is holomorphic at* p*,* $\hat{f}(p) = p$ *and* $\hat{f} : (X, p) \circlearrowleft$ *is rigid.*

In fact, we can give normal forms for the $\hat{f}$ as well. There is also a similar result for polynomial maps near infinity.

9. The valuative tree

As is true throughout much of my work with Favre, the main role is played by the *valuative tree* $\mathcal{V}$. This tree is defined and commented on elsewhere; let us for now simply think of it as encoding local information at the origin in $\mathbf{C}^2$.

When proving dynamic desingularization, the valuative tree enters as follows. First, there is an induced selfmap of the tree. Second, for topological reasons, this selfmap must admit a fixed point, or *eigenvaluation*. Third, the induced dynamics on $\mathcal{V}$ is reminiscent of the dynamics of a holomorphic selfmap on the unit disk in $\mathbf{C}$. In particular, it has some attracting properties. It turns out that what gives the key to dynamic desingularization (and hence to the asymptotic rate of attraction) is a basin of attraction on the valuative tree. The situation is illustrated in Figure 1.

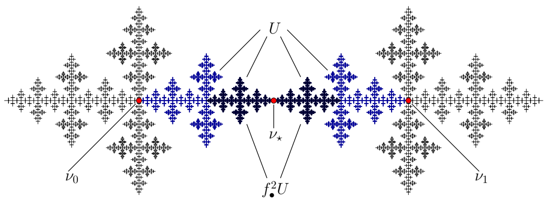

Figure 1. A basin of attraction of an (irrational) eigenvaluation.