Research on general singularities by Mattias Jonsson

Much of my past and current work in mathematics deals with *singularities*, in a broad sense. This research is largely joint with Charles Favre at CNRS, Paris. The main theme is to take a complicated mathematical object and make it simpler through a well-defined procedure. For instance, a curve which is singular (i.e. does not look like a curve on a small scale) can typically be viewed as the "shadow" of one or more smooth curves. Similarly a complicated (discrete) dynamical system can similarly be seen as the "shadow" of a simpler one.

We have found a quite general new method—based on valuations—to study singularities of a variety of objects in a unified way. So far we have worked in a local, two-dimensional setting, but plan to extend our approach: see below. I now outline our contributions to three areas: Algebraic Geometry, Pluripotential Theory and Dynamical Systems.

Algebraic Geometry. The study of singularities forms an integral and classical part of algebraic geometry. While our analysis applies more generally (e.g. over fields other than $\mathbf{C}$), I shall describe it here in the analytic setting.

The fundamental work of Hironaka [H] shows that singularities can be resolved by birational changes of coordinates. For instance, the singular curve $\{y^2 = x^3\}$ is the image of the smooth curve $\{w = 0\}$ under the map $(x, y) = (z^2(1 + w), z^3(1 + w)^2)$. Similarly, ideals, generated by finitely many functions, can be simplified by so called log-resolutions, rendering them locally principal.

However, these resolutions are not in general unique, and in many situations it is desirable to describe singularities in more quantitative terms. Notions such as multiplier ideals and log-canonical threshold serve this purpose; the latter plays an important role in Mori's minimal model program [Mo].

Charles Favre and I use valuations to study singularities. In a way, this method goes back to Zariski, but our approach is more analytic in nature and indeed better suited to deal with objects such as plurisubharmonic functions (see below).

In our study, the main protagonist is what we call the *valuative tree* $\mathcal{V}$. Its elements are normalized valuations on the ring of holomorphic functions at the origin in $\mathbf{C}^2$ and may be thought of as orders of vanishing at the origin. For instance, given local coordinates (x, y) and a real number $t \in [1, \infty]$, one may look at the monomial valuation with weights 1 and t on x and y, respectively: this gives the order of vanishing of a germ at the origin within the cone $\{|x| < r, |y| < |x|^t\}$. Thinking of t as a parameter we obtain an embedding of the interval $[1, \infty]$ in $\mathcal{V}$, and as we show in [FJ1], the full space $\mathcal{V}$ is the union of uncountably many real intervals welded together in a way that no cycles appear. It is hence naturally a tree: more precisely an $\mathbf{R}$-tree with strong self-similar properties: see Figure 1.

The valuative tree $\mathcal{V}$ can be used to encode singularities. Without going into detail, holomorphic functions (or curves) and ideals both give rise to functions on $\mathcal{V}$ with strong concavity properties. We showed in [FJ1] how to define a natural Laplace operator on an $\mathbf{R}$-tree, allowing us to identify such concave functions with positive measures on $\mathcal{V}$.

As a consequence, functions or ideals give rise to (positive atomic) *tree measures* on $\mathcal{V}$. As exploited in [FJ1] and [FJ3], these tree measures efficiently describe singularities. For instance, in the case of an (integrally closed) ideal, the decomposition of the tree measure into atoms recovers Zariski's celebrated factorization into simple complete ideals. Furthermore, multiplier ideals and log-canonical thresholds may be elegantly computed from tree measures, and allowed us to prove a conjecture to the effect that every integrally closed ideal may be realized as a multiplier ideal (this was verified independently by Lipman and Watanabe [LW]).

Pluripotential theory. A plurisubharmonic (psh) function (near the origin in $\mathbf{C}^2$) may be viewed as a generalization of an ideal, i.e. finitely many holomorphic functions, in roughly the same way as a positive measure generalizes a finitely atomic measure.

Several invariants, including Lelong numbers, Kiselman numbers, generalized Lelong numbers, multiplier ideals and log-canonical thresholds, had previously been proposed [D] in order to study singularities of psh functions, but the relations between them remained less than perfectly understood. Using our valuative technique, we are able to relate these invariants in a quite precise way.

A first step in our approach is to study psh functions with logarithmic singularities: these are essentially of the form $u = c \log \sum_1^n |h_j|^2$, for holomorphic functions h_j and a constant $c > 0$. The singularity of this psh function naturally coincides (up to the constant c) with that of the ideal generated by $h_1, \dots, h_n$. In particular, u then has a naturally associated tree measure on $\mathcal{V}$.

The powerful technique of Demailly [DK] shows that a general psh function may be approximated in a strong way by psh functions with logarithmic singularities. This allows us (after some work) to associate tree measures also to psh functions: these are still positive measures of finite mass on the valuative tree $\mathcal{V}$, but need no longer be atomic. Details can be found in [FJ2].

As we show in [FJ2] and in [FJ3], the tree measure contains essentially complete information on the singularity of a psh function u. It allows us to compute all Kiselman numbers, all (generalized) Lelong numbers, all multiplier ideals and the log-canonical threshold of u, and even of all its multiples. Under mild regularity assumptions, we also obtain a formula for the mixed Monge-Ampère mass of two psh functions as a suitably defined bilinear intersection product of their associated tree measures.

Our valuative technique also has had two striking applications. First, in [FJ3] we settled the "openness conjecture" by Demailly and Kollár [DK]: for any psh function u, the set of $c > 0$ such that $\exp(-cu)$ is locally integrable, is an open interval. Second, we showed in [FJ2] that every positive closed (1,1) current T admits a *attenuation of singularities*: there exists a finite composition of blowups such that the pull-back of T decomposes into two parts, the first associated to a curve with simple normal crossing singularities, the second having arbitrarily small Lelong numbers.

Dynamical systems. Our work on singularities was originally motivated by a problem in dynamical systems [6], and lead us to a study of superattracting fixed point germs $f : (\mathbf{C}^2, 0) \circlearrowleft$. Such dynamical systems appear in many numerical algorithms, such as Newton's method, and the convergence rate of the algorithm is governed by the behavior of the iterates f^n of f, specifically of $c(f^n)$, the generic order of vanishing of f^n at the origin.

The sequence $c(f^n)$ is supermultiplicative, hence the limit $c_\infty := \lim c(f^n)^{1/n}$ exists and describes the asymptotic rate of convergence to the origin. We show in [FJ4] that this number is always a quadratic integer and make the interpretation of the limit more precise in proving that there exists a psh function u_∞ with the invariance property $u_\infty \circ f = c_\infty u_\infty$.

Trying to find simple normal forms is a natural task when undertaking the study of a class of dynamical systems, and plays a key role in our approach. While it is probably impossible to find a useful normal form at the origin for a superattracting fixed point germ, we show that every germ can be made *rigid*, i.e. having totally invariant critical set, after performing a finite number of point blowups. Previous work by my collaborator C. Favre then provides quite strong normal forms.

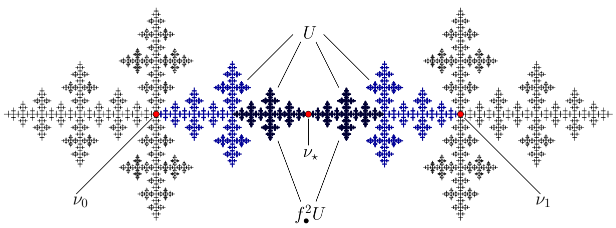

Figure 1: A basin of attraction of an (irrational) eigenvaluation.

To prove that a germ f can be made rigid requires control of the action of f on the space of all possible blowups. But as we showed in [FJ1], the latter space is isomorphic to the valuative tree $\mathcal{V}$. Moreover, we show in [FJ4] that the action of f on $\mathcal{V}$ respects the tree structure in a strong sense. A topological argument then shows the existence of a fixed point—or *eigenvaluation*—for this action. Rigidity results from an analysis of the basin of attraction of the eigenvaluation: see Figure 1.

Most of our analysis is also valid for (not necessarily proper) polynomial mappings $F : \mathbf{C}^2 \circlearrowleft$, where the line at infinity plays the role of the (blowup of the) origin in the local case. Due to the possible presence of indeterminacy points at infinity, the analysis is more delicate, but we show, among other things, that the asymptotic degree $d_\infty := \lim \deg(F^n)^{1/n}$ is a quadratic integer.

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