

Singularities in pluripotential theory

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1. INTRODUCTION

These notes aim at giving a brief introduction to some of my own research on singularities in pluripotential theory. They do *not* try to give a general overview of the subject and certainly do not give credit to all researchers in the field. Use them at your own risk!

2. CURRENTS AND PLURISUBHARMONIC FUNCTIONS

For the purposes of these notes, we will work locally in two dimensions, i.e. in a neighborhood of the origin in \mathbf{C}^2 . This simplifies part of the discussion. More importantly, there are as of today hurdles to pass before generalizing my own contributions from two to even three dimensions.

By definition, a plurisubharmonic (psh) function on $(\mathbf{C}^2, 0)$ is an upper semi-continuous (usc) function, defined in a neighborhood of the origin in \mathbf{C}^2 , whose restriction to any complex line is subharmonic. A positive closed current is an object of the form $dd^c u = \frac{i}{2\pi} \partial\bar{\partial}u$, where u is psh.

For the discussion that follows, it is instructive to think of psh functions or positive closed currents as (proportional to) limits of curves, or ideals in \mathcal{O}_0 , the ring of holomorphic germs at the origin in \mathbf{C}^2 . Indeed, to each holomorphic germ $\phi \in \mathcal{O}_0$ we may associate a psh function $u = \log |\phi|$. Similarly, if I is an ideal in \mathcal{O}_0 generated by ϕ_1, \dots, ϕ_k , then we may associate to I either the psh function $u = \frac{1}{2} \log \sum |\phi_i|^2$ or the psh function $u = \log \max_i |\phi_i|$. The choice of generators ϕ_i influences these functions only very mildly. Alternatively, to a curve C in $(\mathbf{C}^2, 0)$ we may associate the *current of integration* $[C]$.

By taking limits of multiples of the preceding objects we obtain all psh functions or currents on $(\mathbf{C}^2, 0)$. For instance, given $t > 1$ irrational, we can by choosing the integers p_n and q_n appropriately, make the psh functions $u_n = q_n^{-1} \log \max\{|y|^{q_n}, |x|^{p_n}\}$ converge to the psh function $u_\infty = \log \max\{|y|, |x|^t\}$.

We can also construct interesting currents as follows. Let Σ denote the product space $\{+1, -1\}^{\mathbf{N}}$. To any sequence $\sigma = (\sigma_i)_1^\infty \in \Sigma$, associate a smooth curve $C_\sigma := \{y = f_\sigma(x)\}$, where $f_\sigma(x) = \sum_1^\infty \sigma_i x^i$. Let ρ be the “equilibrium measure”

on Σ , i.e. the product measure arising from the uniform distribution on $\{+1, -1\}$. Then the current

$$T := \int_{\Sigma} [C_{\sigma}] d\rho(\sigma)$$

can be interpreted as the current of integration on a ‘‘Cantor bouquet’’ of curves.

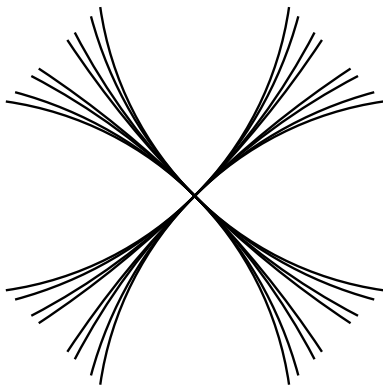


FIGURE 1. A Cantor bouquet of curves

3. LELONG NUMBERS AND ATTENUATION OF SINGULARITIES

We say that a psh function u on $(\mathbf{C}^2, 0)$ (or the associated current $dd^c u$) has a *singularity* at the origin if $u(0) = -\infty$. The ‘‘strength’’ of this singularity varies with u . Two good examples of ‘‘strong’’ singularities to keep in mind are $u = \log \|p\|$ and $u = \log |y|$. An example of a ‘‘weak’’ singularity is given by $u = -\log(-\log \|p\|)$, which tends to $-\infty$ quite slowly.

The first and most important measurement of the singularity is given by the *Lelong number* $\nu^L(u)$, defined as

$$\nu^L(u) = \lim_{r \rightarrow 0} \frac{1}{\log r} \sup_{B(0,r)} u.$$

When $u = \log |\phi|$ for ϕ a holomorphic germ, $\nu^L(u)$ is exactly the multiplicity $m(\phi)$. Equivalently, if C is a curve, then the Lelong number of the current of integration $[C]$ equals the multiplicity $m(C)$ of C .

Since currents generalize curves, one may ask whether the theorem on resolution of curve singularities can be generalized to currents. In the strongest form, the answer is no. Indeed consider the Cantor bouquet current illustrated in Figure 1. In a suitable sense, the current defines exactly two tangent vectors at the

origin in \mathbf{C}^2 . However, it should not be viewed as a normal crossings singularity as each tangent direction is comprised of several curves that are tangent to each other.

After blowing up the origin, $\pi : X \rightarrow (\mathbf{C}^2, 0)$, the current decomposes into two parts. By the Cantor structure, the strict transform of the current consists of two disjoint copies of the current. The total transform will in addition contain the current of integration $[E]$ on the exceptional divisor $E = \pi^{-1}(0)$. The situation is illustrated in Figure 2. By further blowups we may decompose the current into

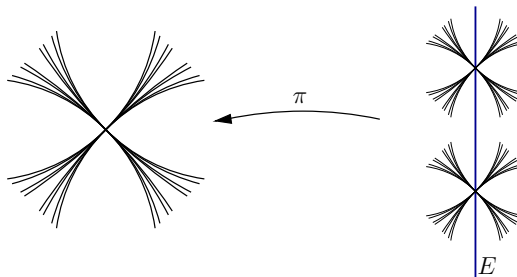


FIGURE 2. Blowing up the Cantor bouquet

many small copies of the original current. In particular we will never arrive at a normal crossings situation. However, the copies are indeed small in the sense that their Lelong numbers can be made arbitrarily small. In fact, Favre and I proved that such an approximate desingularization, or *attenuation of singularities* is possible in general.

Theorem 3.1. *For any positive closed current T on $(\mathbf{C}^2, 0)$ there exists given any $\epsilon > 0$, a finite composition $\pi : X \rightarrow (\mathbf{C}^2, 0)$ of point blowups such that the total transform π^*T of T can be written as $\pi^*T = S_1 + S_2$, where S_1 is associated to a normal crossings divisor and the Lelong number of S_2 is at most ϵ at any point on the exceptional divisor $\pi^{-1}(0)$.*

4. SINGULARITY EXPONENTS AND THE OPENNESS CONJECTURE

The definition of singularity exponent or Arnold multiplicity can be adapted to psh functions or currents. Namely, if u is psh, then we define the *singularity exponent* of u as $c(u)$, where

$$c(u) := \sup\{c > 0 \mid \exp(-2cu) \in L_{\text{loc}}^2\}.$$

The reciprocal $\lambda(u)$ known as the *Arnold multiplicity* of u is often more natural. By definition:

$$\lambda(u) := \inf\{\lambda > 0 \mid \exp(-2u/\lambda) \in L_{\text{loc}}^2\}.$$

Notice that $\lambda(tu) = t\lambda(u)$ for any psh function u and any $t > 0$.

It is a natural question whether integrability holds at the critical exponent. Demailly and Kollár conjectured that the answer is no. This was proved by Favre and myself.

Theorem 4.1. *For any psh function u on $(\mathbf{C}^2, 0)$ the function $\exp(-2c(u)u)$ is not locally integrable at the origin.*

The corresponding question in dimension at least three is still open. Hironaka's theorem implies that integrability fails if u has *logarithmic singularities* in the following sense:

$$u = t \log \sum_1^n |\phi_i|^2 + O(1),$$

where $t > 0$ and ϕ_i are holomorphic germs.

5. THE VALUATIVE TREE AND TREE MEASURES

The main tool that Favre and I use for the study of singularities is *the valuative tree* \mathcal{V} . See Figure 3

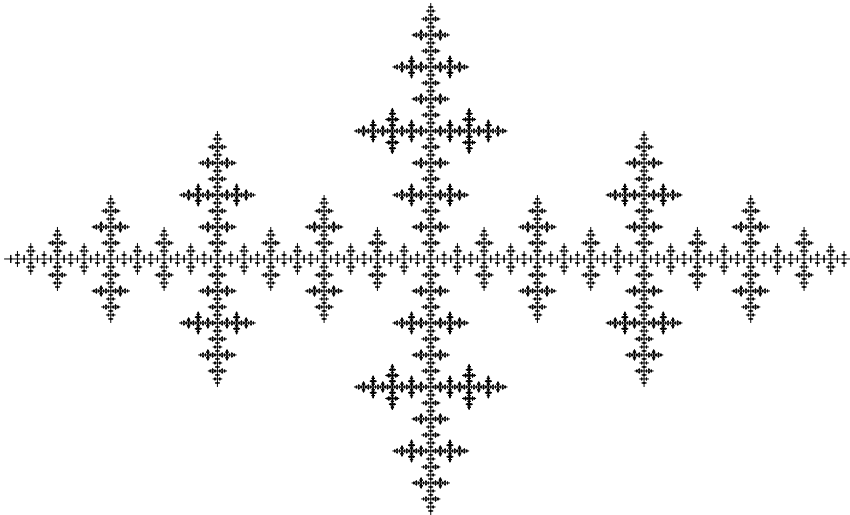


FIGURE 3. The valuative tree. This rough picture emphasizes the self-similar structure but does not explain how the segments arise.

As described elsewhere, we may encode the singularities of holomorphic germs and ideals in \mathcal{O}_0 in terms of positive atomic measures on the valuative tree.

This generalizes to currents, whose singularities can be encoded by more general positive measures on \mathcal{V} , the *tree measure* of the current.

The proof of attenuation of singularities essentially reduces to finding a partition of the valuate tree such that each element has small mass for the tree measure. This is also roughly the idea for the openness conjecture.