The quasiconformal Jacobian problem

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1 Introduction.

Which nonnegative functions can arise, up to a bounded multiplicative error, as Jacobian determinants $J_f(x) = \det(Df(x))$ of quasiconformal mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$? Which metric spaces are bi-Lipschitz equivalent to $\mathbb{R}^n$, $n \geq 2$? We first survey what is known about these problems, and point out how it follows from recent work of B. Kleiner and the first author [BoK] that the two problems are in fact equivalent in dimension $n = 2$. Then we proceed to exhibit a hitherto unknown class of functions which satisfy the requirement of the first question (also in dimension $n = 2$). To be more specific, we exploit the connection between the two problems, and deduce the following result from a theorem of Fu [F].

Theorem 1.1 *Let u be a locally integrable function in $\mathbb{R}^2$ with distributional gradient ∇u in $L^2(\mathbb{R}^2)$. Then there exists a quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ together with a constant $C \geq 1$ such that*

$$\frac{1}{C} e^{2u(x)} \leq J_f(x) \leq C e^{2u(x)} \quad \text{for a.e.} \quad x \in \mathbb{R}^2. \tag{1.2}$$

The dilatation of f and the constant C depend only on the L^2-norm of ∇u.

We recall here that a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is *quasiconformal* if f belongs to the local Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ and if there exists $K \geq 1$ such that

$$|Df(x)|^n \leq K J_f(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n. \tag{1.3}$$

Here $Df(x) = (\partial_i f_j(x))$ is the formal differential matrix of f. The smallest K for which (1.3) holds is called the *dilatation* of f. We refer to [BI], [Res], and [V] for the basic theory of quasiconformal mappings in $\mathbb{R}^n$.

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We shall discuss the merits and flaws of Theorem 1.1 in more detail in Section 4.

Recently, Burago and Kleiner [BuK], and McMullen [M] exhibited functions w on $\mathbb{R}^n$ such that both w and w^{-1} belong to $L^\infty(\mathbb{R}^n)$ but there is no bi-Lipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with

$$J_f(x) = w(x) \quad \text{for a.e. } \ x \in \mathbb{R}^n. \tag{1.4}$$

Since every quasiconformal mapping with essentially bounded Jacobian is Lipschitz, it follows that equation (1.4) cannot hold for quasiconformal mappings either.

In 1990, David and Semmes [DS1] formulated our opening question, which asks for *comparability* instead of equality as in (1.4). To make this version more precise, let us agree to call each nonnegative locally integrable function in $\mathbb{R}^n$ a *weight*. We coin the following question the *quasiconformal Jacobian problem* in $\mathbb{R}^n$: for which weights w in $\mathbb{R}^n$, $n \geq 2$, do there exist a quasiconformal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and a constant $C \geq 1$ such that

$$\frac{1}{C} w(x) \leq J_f(x) \leq C w(x) \ \text{ for a.e. } \ x \in \mathbb{R}^n. \tag{1.5}$$

This question of David and Semmes appears still difficult, especially in dimensions $n \geq 3$. Note that the answer for the weights w considered by Burago, Kleiner, and McMullen is trivially affirmative: the identity map $f(x) = x$ satisfies (1.5). In a way, the quasiconformal Jacobian problem asks for a characterization of Jacobian determinants of quasiconformal mappings up to bi-Lipschitz mappings.

If a weight w satisfies (1.5) for some $C \geq 1$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ quasiconformal, we say that w is *comparable to a quasiconformal Jacobian*.

In search for a characterization, one should obviously begin with the known necessary conditions. In our case, these are not multifarious. According to the well-known result of Gehring [G], the Jacobian of a quasiconformal mapping is (in modern parlance) an A_∞-*weight*. By definition, these are weights w for which there exist $\varepsilon > 0$ and $C \geq 1$ such that

$$\left(⨍_B w^{1+\varepsilon}(x)\, dm_n(x) \right)^{1/(1+\varepsilon)} \leq C ⨍_B w(x)\, dm_n(x) \tag{1.6}$$

for every ball $B \subset \mathbb{R}^n$. Here and in what follows, the barred integral sign denotes mean value:

$$u_E := ⨍_E u(x)\, dm_n(x) := \frac{1}{m_n(E)} \int_E u(x)\, dm_n(x), \quad u \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{1.7}$$

where $m_n(E) = \int_E dm_n$ is the Lebesgue n-measure of a bounded measurable set $E \subset \mathbb{R}^n$ (assumed positive in (1.7)). We write $A_\infty(\mathbb{R}^n)$ for the collection of all A_∞-weights in $\mathbb{R}^n$.

It follows from the A_∞-condition that $\log(J_f) \in \mathrm{BMO}(\mathbb{R}^n)$ for every quasiconformal Jacobian, as observed by Reimann [Rei]. Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of those locally integrable functions u in $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that

$$⨍_B |u(x) - u_B|\, dm_n(x) \leq C \tag{1.8}$$

for every ball $B \subset \mathbb{R}^n$. Indeed, we have the well-known implications:

$$w \in A_\infty(\mathbb{R}^n) \Rightarrow \log(w) \in \mathrm{BMO}(\mathbb{R}^n), \tag{1.9}$$

and

$$u \in \mathrm{BMO}(\mathbb{R}^n) \Rightarrow e^{\delta u} \in A_\infty(\mathbb{R}^n), \tag{1.10}$$

where $\delta = \delta(n, C) > 0$ is a positive constant depending only on n and on C as in (1.8). For these and other basic properties of A_∞- and BMO-functions see [RR], [St].

Not every A_∞-weight can be comparable to a quasiconformal Jacobian. For example, it is easy to see from the defining inequality (1.3) that the A_∞-weight $w(x) = \mathrm{dist}(x, L)$ cannot have this property, where $L \subset \mathbb{R}^n$ is a straight line; else the hypothetical mapping would collapse the line to a point. The obvious problem with this weight is that w vanishes (continuously) on L, which is in particular a locally rectifiable curve admitting line integration. To illustrate the subtlety of the problem, consider a quasiconformal mapping $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{1}{C}|x-y|^\alpha \le |g(x) - g(y)| \le C|x-y|^\alpha \tag{1.11}$$

for each pair of points $x, y \in \mathbb{R} \subset \mathbb{R}^2$, and for some constants $C \ge 1$ and $\frac{1}{2} < \alpha < 1$ independent of the points. Thus g is a "snowflake mapping", taking the real line to a fractal curve Γ in a uniformly expanding fashion. Moreover, g can be made C^∞ outside $\mathbb{R}$. For the construction of such a map, see [T]. Now set $f = g^{-1}$. Then it is easy to see that the Jacobian J_f is continuous in $\mathbb{R}^2$ and vanishes on the curve Γ.

We refer the reader to the paper [S1] by Semmes for an informative discussion on the above and other similar examples, and on related matters.

In trying to identify the subclass of A_∞-weights that are comparable to quasiconformal Jacobians, David and Semmes [DS1], [S1] introduced the class of "strong A_∞-weights". In later literature, the term "metric doubling measure" for the associated Borel measure has become more popular [DS2], [S4], [S5], [H]. To describe these concepts, consider a doubling Borel measure μ on $\mathbb{R}^n$, where by *doubling* we mean that there exists a constant $C \ge 1$ such that

$$\mu(B(x, 2r)) \le C\mu(B(x, r)) \tag{1.12}$$

for every (open) ball $B(x, r) \subset \mathbb{R}^n$ of center x and radius $r > 0$. We also assume that μ is nontrivial. Associated with each such measure is a quasimetric

$$d_\mu(x, y) := \mu(B_{xy})^{1/n}, \tag{1.13}$$

where $B_{xy} = B(x, |x-y|) \cup B(y, |x-y|)$. That is, $d_\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is symmetric, vanishes exactly on the diagonal, and satisfies

$$d_\mu(x, y) \le K(d_\mu(x, z) + d_\mu(z, y)) \tag{1.14}$$

for every triple x, y, z of points in $\mathbb{R}^n$, for some constant $K \ge 1$ independent of the triple.

If (1.14) holds with $K = 1$, then the quasimetric is a metric. Following David and Semmes, we call μ a *metric doubling measure* if d_μ is comparable to a metric; that is, if there is a metric δ_μ on $\mathbb{R}^n$ and a constant $C \geq 1$ such that

$$\frac{1}{C}\delta_\mu(x,y) \leq d_\mu(x,y) \leq C\delta_\mu(x,y) \tag{1.15}$$

for each pair of points $x, y \in \mathbb{R}^n$.

The pushforward $\mu = f_* m_n$ of the Lebesgue measure m_n in $\mathbb{R}^n$ under a quasiconformal mapping f is a metric doubling measure. This follows from the change of variables formula and the basic distortion theorems for quasiconformal mappings. Indeed, we can take

$$\delta_\mu(x,y) := |f(x) - f(y)|, \qquad x, y \in \mathbb{R}^n; \tag{1.16}$$

for then

$$\delta_\mu(x,y) \simeq \operatorname{diam}(f(B_{xy})) \simeq m_n(f(B_{xy}))^{1/n} = \left(\int_{B_{xy}} J_f(x)\, dm_n(x)\right)^{1/n}, \tag{1.17}$$

where the constants of comparability depend only on n and the dilatation of f.

Conversely, David and Semmes showed in [DS1] that every metric doubling measure μ in $\mathbb{R}^n$ has an A_∞-density,

$$d\mu(x) = w(x)\, dm_n(x), \tag{1.18}$$

where $w \in A_\infty(\mathbb{R}^n)$ with A_∞-data depending only on the dimension n and the data associated with μ. They called a density w of a metric doubling measure as in (1.18) a *strong* A_∞*-weight*, and asked if every such weight was comparable to a quasiconformal Jacobian.

The answer to this latter question turned out to be *no*. This was shown a few years later by Semmes [S2] in dimensions $n \geq 3$, and recently by Laakso [L] in dimension $n = 2$. Currently, there seems to be no good guess as to what analytic conditions would characterize quasiconformal Jacobians. On the other hand, there are some interesting classes of weights that are not known to be comparable to quasiconformal Jacobians, nor is there a counterexample available. We shall discuss open problems along these lines in Setion 4. Finally, we remark that a nontrivial sufficient condition of a geometric character for a weight to be comparable to a quasiconformal Jacobian was given in [BHR].

Let us now discuss the relationship between the two problems posed in the beginning of this article. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping, then it follows from the definitions (1.15), (1.16), and (1.17), that

$$f\colon \mathbb{R}^n \to (\mathbb{R}^n, \delta'_\mu)$$

is a bi-Lipschitz homeomorphism for every metric δ'_μ in $\mathbb{R}^n$ that is comparable to the quasimetric d_μ associated with the metric doubling measure $\mu = f_* m_n$.

Conversely, if μ is a metric doubling measure with a metric δ_μ as in (1.15), and if

$$f\colon \mathbb{R}^n \to (\mathbb{R}^n, \delta_\mu)$$

is a bi-Lipschitz homeomorphism, then $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal and the strong A_∞-weight of (1.18) is comparable to J_f; that is, formula (1.5) is valid, and the dilatation of f and the constant $C \geq 1$ in (1.5) only depend on the data associated with μ. These claims are easily verified by using the quasisymmetry property (see (1.20) below) of a quasiconformal mapping; see [S1], [S4, B.19], [H, 14.14].

We now make the further informal assertion that *in dimension* $n = 2$ *the two problems posed in the beginning of this article are equivalent*.

This assertion should be understood as follows. Suppose first that the second problem is completely understood, and that we are given a weight w in $\mathbb{R}^2$. To see whether w is comparable to a quasiconformal Jacobian, we perform two tests: is the measure determined by

$$d\mu = w\, dm_2$$

a metric doubling measure, and is the associated (quasi)metric space $(\mathbb{R}^2, d_\mu)$ bi-Lipschitz equivalent to $\mathbb{R}^2$. As explained above, the comparability is equivalent to having an affirmative answer to both tests. Note that here we abuse terminology and speak of bi-Lipschitz mappings between the quasimetric space $(\mathbb{R}^2, d_\mu)$ and $\mathbb{R}^2$; the more rigorous language would involve a metric δ_μ as in (1.15).

Next, suppose that the first problem is completely understood, and that we are given a metric space (X, d). To see whether X is bi-Lipschitz equivalent to $\mathbb{R}^2$, we again perform some obvious tests. First, X should have Hausdorff dimension two and the Hausdorff 2-measure $\mathcal{H}^2$ in X should satisfy the condition of *Ahlfors* 2*-regularity*: there is a constant $C \geq 1$ such that

$$\frac{1}{C} r^2 \leq \mathcal{H}^2(B(x,r)) \leq Cr^2 \tag{1.19}$$

for each $x \in X$ and $0 < r < \infty$. Second, X should satisfy the condition of *linear local connectivity*: there is a constant $C \geq 1$ such that every pair of points $a, b \in B(x, r)$ can be joined by a continuum in $B(x, Cr)$ and every pair of points $a, b \in X\backslash B(x, r)$ can be joined by a continuum in $X\backslash B(x, r/C)$.

Kleiner and the first author have shown that if a complete metric space (X, d) homeomorphic to $\mathbb{R}^2$ is Ahlfors 2-regular and linearly locally connected, then it is quasisymmetrically equivalent to $\mathbb{R}^2$. That is, there is a homeomorphism

$$h\colon X \to \mathbb{R}^2$$

satisfying the following *quasisymmetry* condition:

$$d(a, b) \leq t d(a, c) \ \text{ implies } \ |h(a) - h(b)| \leq \eta(t)|h(a) - h(c)|, \tag{1.20}$$

whenever a, b, c is a triple of distinct points in X and $t > 0$. Here $\eta\colon (0, \infty) \to (0, \infty)$ is a function, independent of the triple, with $\eta(t) \to 0$ as $t \to 0$. Just as in (1.16) and (1.17), we see that the pushforward $\mu = f_* \mathcal{H}^2$ of the Hausdorff measure $\mathcal{H}^2$ is a metric doubling measure in $\mathbb{R}^2$. Hence by the David-Semmes theorem it has a (strong) A_∞-density w as in (1.18). Moreover, (X, d) is bi-Lipschitz equivalent to $(\mathbb{R}^2, d_\mu)$, where d_μ is given in (1.13) (with the usual abuse of terminology).

Now if w is comparable to a quasiconformal Jacobian, then we have the bi-Lipschitz equivalence

$$\mathbb{R}^2 \simeq (\mathbb{R}^2, d_\mu) \simeq (X, d).$$

While conversely, if w is not a weight that is comparable to a quasiconformal Jacobian, the bi-Lipschitz equivalence between $\mathbb{R}^2$ and $(\mathbb{R}^2, d_\mu)$, and hence between $\mathbb{R}^2$ and X, must fail by the preceding discussion.

We remark that while the passage from the second question to the first question is equally valid in each dimension $n \geq 2$ (with a similar argument), the equivalence of the two questions fails if $n \geq 3$. This is because there are no results along the lines of [BK] in dimensions $n \geq 3$; this was shown by Semmes in [S3].

In sum, one can say that in general the quasiconformal Jacobian problem is an "easier" problem than recognizing $\mathbb{R}^n$ up to a bi-Lipschitz equivalence. In dimension $n = 2$, the issues are equivalent.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we show that weights of the form

$$w = e^{nu}, \qquad u \in L^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n), \quad 0 < \alpha < n, \tag{1.21}$$

are strong A_∞-weights in each $\mathbb{R}^n$, $n \geq 2$, where the function space $L^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$ consists of all functions u with "α derivatives" in $L^{\frac{n}{\alpha}}(\mathbb{R}^n)$. For a precise definition, see Section 3.

We do not know whether weights of the form (1.21) for $0 < \alpha < 1$ and $n = 2$, and for $0 < \alpha < n$ and $n \geq 3$, are comparable to quasiconformal Jacobians. This and other open problems will be discussed in Section 4.

2 Proof of Theorem 1.1.

We begin by citing the following theorem of Fu [F]:

Theorem 2.1 *Let X be a complete Riemannian 2-manifold that is homeomorphic to $\mathbb{R}^2$. There are absolute constants $\varepsilon_0 > 0$ and $L_0 > 0$ with the following property: if the integral curvature of X does not exceed ε_0, then X is bi-Lipschitz equivalent to $\mathbb{R}^2$ with bi-Lipschitz constant L_0.*

In fact, by recent work of Lang and the first author [BoL], one can choose ε_0 to be any number less that 2π in Theorem 2.1 (then we have $L_0 = L_0(\varepsilon_0)$).

We shall prove Theorem 1.1 by reducing it to Fu's theorem. First we require a lemma (which may well be known, but we do not have a reference). We use the usual notation $\| \cdot \|_p$ for the L^p-norm of a function, where $1 \leq p \leq \infty$.

Lemma 2.2 *Let $u\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a smooth function with compact support. Then for every $\varepsilon > 0$ there exists a decomposition $u = s + b$ into two compactly supported smooth functions such that*

$$\|\Delta s\|_1 \leq \varepsilon \tag{2.3}$$

and

$$\|b\|_\infty \leq \frac{13}{\varepsilon}\|\nabla u\|_2^2. \tag{2.4}$$

Proof: Assume first that $u\colon \mathbb{R}^n \to \mathbb{R}$ is continuous, nonnegative, compactly supported, smooth in the open set $\Omega = \{x \in \mathbb{R}^n : u(x) > 0\}$, and assume that the distributional gradient ∇u belongs to $L^2(\mathbb{R}^n)$. We shall then show that for each $\varepsilon > 0$ there exists a decomposition $u = s + b$ into two compactly supported continuous functions such that

$$\|\Delta s\|_1 \leq \varepsilon \tag{2.5}$$

and that

$$\|b\|_\infty \leq \frac{6}{\varepsilon}\|\nabla u\|_2^2, \tag{2.6}$$

where the distribution Δs is a signed measure and $\|\Delta s\|_1$ denotes its total variation. Later we shall show how the assertion of the lemma can be reduced to this case.

Thus, fix $\varepsilon > 0$. We may assume $\|\nabla u\|_2 > 0$. Using Sard's theorem, we can find a constant $L > 0$ with

$$\frac{2}{\varepsilon}\|\nabla u\|_2^2 < L < \frac{3}{\varepsilon}\|\nabla u\|_2^2$$

such that kL for $k = 1, 2, \ldots$ is a regular value of u. This implies that $\max(u) \neq kL$, $k = 1, 2, \ldots$, and that each each level set

$$\Gamma_k := \{x \in \mathbb{R}^n : u(x) = kL\}, \quad k = 1, 2, \ldots,$$

is either empty or a smooth codimension 1 submanifold of $\mathbb{R}^n$. For $k = 1, 2, \ldots$, let

$$H_k := \{x \in \mathbb{R}^n : u(x) < kL\},$$

and

$$\Omega_k := \{x \in \mathbb{R}^n : kL < u(x) < (k+1)L\}.$$

Then $\partial H_k = \Gamma_k$, $\partial \Omega_k = \Gamma_k \cup \Gamma_{k+1}$, and $\Omega_k = H_{k+1} \setminus \overline{H}_k$. Let $N \in \mathbb{N}_0$ be the smallest integer such that $u < (N+1)L$. Then $H_{N+1} = \mathbb{R}^n$ and $\Omega_{N+1} = \emptyset$.

If $N = 0$, then $0 \leq u < L$, and so we can take $s = 0$ and $b = u$. So we may assume $N \geq 1$ in the following.

Since each open set Ω_k is regular for the Dirichlet problem for $1 \leq k \leq N$, we can solve the Dirichlet problem on Ω_k with boundary values $u|\partial\Omega_k$, and obtain a function v_k, harmonic in Ω_k and continuous in the closure $\overline{\Omega}_k$. We define

$$\hat{s}(x) = \begin{cases} v_k(x), & x \in \overline{\Omega}_k \ \text{ for } \ 1 \leq k \leq N, \\ L, & x \in H_1. \end{cases}$$

Then $\hat{s}$ is well-defined and continuous, and $s := \hat{s} - L$ is compactly supported. We show that (2.5) and (2.6) hold for s and $b := u - s$.

First observe that

$$\|b\|_\infty = \|u - s\|_\infty \leq 2L < \frac{6}{\varepsilon}\, \|\nabla u\|_2^2$$

by construction and by the maximum principle; so (2.6) indeed holds. It is estimate (2.5) that calls for a proof. To this end, note that the distribution $\Delta s = \Delta \hat{s}$ is supported on $\partial\Omega_1 \cup \ldots \cup \partial\Omega_N$. To deal with each boundary term at a time, we let

$$s_k(x) = \begin{cases} 0, & x \in H_k, \\ v_k - kL, & x \in \overline{\Omega}_k, \\ L, & x \in \mathbb{R}^n \setminus H_{k+1}. \end{cases}$$

for $1 \leq k \leq N$. Then

$$s = \sum_{k=1}^{N} s_k,$$

and so

$$\Delta s = \sum_{k=1}^{N} \Delta s_k.$$

The measure Δs_k is supported on $\Gamma_k \cup \Gamma_{k+1}$. Since the function $s_k|H_{k+1}$ is subharmonic, while $s_k|\mathbb{R}^n \setminus \overline{H}_k$ is superharmonic, it follows that $\Delta s_k = \mu_k^+ - \mu_k^-$ for nonnegative measures μ_k^+ and μ_k^- supported on Γ_k and Γ_{k+1}, respectively.

We have that

$$\begin{aligned} \mu_k^-(\mathbb{R}^n) = \mu_k^-(\Gamma_{k+1}) &= \int_{\mathbb{R}^n} \frac{s_k}{L}\, d\mu_k^- = \int_{\mathbb{R}^n} \nabla s_k \cdot \nabla\left(\frac{s_k}{L}\right) dm_n \\ &= \frac{1}{L} \int_{\Omega_k} |\nabla v_k|^2\, dm_n \leq \frac{1}{L} \int_{\Omega_k} |\nabla u|^2\, dm_n \end{aligned}$$

and hence that

$$\sum_{k=1}^{N} \mu_k^-(\mathbb{R}^n) \leq \frac{1}{L} \int |\nabla u|^2\, dm_n. \tag{2.7}$$

We next show that a similar estimate holds for the measures μ_k^+. The function $t_k := L - s_k$ satisfies $t_k = 0$ on $\partial H_{k+1} = \Gamma_{k+1}$ and $t_k = L$ on Γ_k. Moreover, for each nonnegative test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on Γ_k we have that

$$\begin{aligned} \mu_k^+(\mathbb{R}^n) = \mu_k^+(\Gamma_k) &= \frac{1}{L} \int_{\mathbb{R}^n} \varphi t_k\, d\mu_k^+ \\ &= -\frac{1}{L} \int_{\mathbb{R}^n} \nabla s_k \cdot \nabla(\varphi t_k)\, dm_n \\ &= -\frac{1}{L} \int_{\Omega_k} \nabla s_k \cdot \nabla(\varphi t_k)\, dm_n. \end{aligned}$$

Since $\overline{\Omega}_k$ is compact, we can choose φ such that $\varphi \equiv 1$ on $\overline{\Omega}_k$, and obtain

$$\begin{aligned} \mu_k^+(\mathbb{R}^n) &= -\frac{1}{L}\int_{\Omega_k} \nabla s_k \cdot (t_k \nabla\varphi + \varphi\nabla t_k)\, dm_n \\ &= \frac{1}{L}\int_{\Omega_k} |\nabla s_k|^2\, dm_n \le \frac{1}{L}\int_{\Omega_k} |\nabla u|^2\, dm_n. \end{aligned}$$

By summing over k, we obtain

$$\sum_{k=1}^N \mu_k^+(\mathbb{R}^n) \le \frac{1}{L}\int_{\mathbb{R}^n} |\nabla u|^2\, dm_n,$$

which combined with (2.7) yields

$$\|\Delta s\|_1 \le \sum_{k=1}^N \mu_k^+(\mathbb{R}^n) + \sum_{k=1}^N \mu_k^-(\mathbb{R}^n) \le \frac{2}{L}\int_{\mathbb{R}^n} |\nabla u|^2\, dm_n \le \varepsilon$$

as required.

To complete the proof, let $u \in C_0^\infty(\mathbb{R}^n)$ and let $\varepsilon > 0$. We write $u = u^+ - u^-$, and apply the first part of the proof to both u^+ and u^- to obtain (with obvious notation)

$$u = b^+ - b^- + s^+ - s^- \tag{2.8}$$

with

$$\|\Delta(s^+ - s^-)\|_1 \le \|\Delta s^+\|_1 + \|\Delta s^-\|_1 \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and

$$\begin{aligned} \|b^+ - b^-\|_\infty &\le \|b^+\|_\infty + \|b^-\|_\infty \\ &\le \frac{12}{\varepsilon}\|\nabla u^+\|_2^2 + \frac{12}{\varepsilon}\|\nabla u^-\|_2^2 = \frac{12}{\varepsilon}\|\nabla u\|_2^2. \end{aligned}$$

This proves the desired estimates (2.3) and (2.4) except for the smoothness of the decomposition.

To fix this last piece of the proof, we mollify u to obtain

$$u_\delta = \varphi_\delta * u, \qquad \delta > 0,$$

where as usual $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ is a convolving kernel for some $\varphi \in C_0^\infty(\mathbb{R}^n)$ of unit total integral. With the notation as in (2.8), we have that

$$\|\varphi_\delta * (b^+ - b^-)\|_\infty \le \|(b^+ - b^-)\|_\infty$$

and that

$$\|\Delta\left(\varphi_\delta * (s^+ - s^-)\right)\|_1 = \|\varphi_\delta * \Delta(s^+ - s^-)\|_1 \le \|\Delta(s^+ - s^-)\|_1.$$

Since

$$u = u - u_\delta + (b^+ - b^-)_\delta + (s^+ - s^-)_\delta,$$

and since $\|u - u_\delta\|_\infty \to 0$ as $\delta \to 0$, we can choose δ so small that

$$\|u - u_\delta + (b^+ - b^-)_\delta\|_\infty \leq \frac{13}{\varepsilon}\ \|\nabla u\|_2^2.$$

The proof of the lemma is thereby complete. □

Remarks 2.9 (a) Define

$$L^{1,2}(\mathbb{R}^n) = \left\{u \in L^1_{\text{loc}}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\right\}.$$

If $u \in L^{1,2}(\mathbb{R}^n)$, then there exist a constant $c \in \mathbb{R}$ and a sequence (u_i) in $C_0^\infty(\mathbb{R}^n)$ such that $u_i \to u - c$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ and $\nabla u_i \to \nabla u$ in $L^2(\mathbb{R}^n)$; for $n = 2$ one can choose $c = 0$. See [MZ, p. 46, Theorem 1.78 and Lemma 1.83]. Using this fact and some routine arguments, Lemma 2.2 can be extended to nonsmooth functions $u \in L^{1,2}(\mathbb{R}^n)$. We omit the details, since we do not need this more general result.

(b) The function s in the decomposition provided by Lemma 2.2 can be chosen to be the Green potential of a signed measure μ, up to an additive constant. Indeed, $s = G * \mu + h$ for some harmonic function h, where $G(x) = \log(|x|)$ for $n = 2$ and $G(x) = |x|^{2-n}$ for $n \geq 3$. Since s is bounded near infinity, h must be constant. So if we denote the space of signed measures on $\mathbb{R}^n$ by $\mathcal{M}(\mathbb{R}^n)$, then we can record the formal inclusion

$$L^{1,2}(\mathbb{R}^n) \subset G * \mathcal{M}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n) + \mathbb{R}$$

with appropriate norm bounds.

We now proceed to the proof of Theorem 1.1.

We assume first that $u \in C_0^\infty(\mathbb{R}^2)$, and derive an a priori bound for the constant in the claim. The approximation argument required to handle the general case will be given afterwards.

Thus, we consider the Riemannian 2-manifold $X_u = (\mathbb{R}^2,\ e^{u(x)}|dx|)$, that is a conformal deformation of $\mathbb{R}^2$ by the weight e^u for a given function $u \in C_0^\infty(\mathbb{R}^2)$. If we replace u by $u - b$, where $b \in C_0^\infty(\mathbb{R}^2)$, we have that $X_{u-b} = (\mathbb{R}^2,\ e^{u(x)-b(x)}|dx|)$ and X_u are bi-Lipschitz equivalent by the identity map with bi-Lipschitz constant

$$L = \exp(\|b\|_\infty).$$

We apply Lemma 2.2 and write $u = s + b$, where the decomposition is chosen such that

$$\|\Delta s\|_1 \leq \varepsilon_0,$$

where $\varepsilon_0 > 0$ is the absolute constant in Fu's Theorem 2.1. Since the Gaussian curvature of the space $X_u = (\mathbb{R}^2,\ e^{s(x)}|dx|)$ is

$$\kappa = -e^{-2s}\Delta s,$$

we find that

$$\int_{X_s} |\kappa|\, dV_s = \int_{\mathbb{R}^2} |\Delta s|\, dm_2 \le \varepsilon_0.$$

Here $dV_s = e^{2s(x)} dm_2(x)$ is the volume element on X_s.

We conclude that X_s is L_0-bi-Lipschitz equivalent to $\mathbb{R}^2$, and hence that X_u is L-bi-Lipschitz equivalent to $\mathbb{R}^2$, where

$$L = L_0 \exp\left(\frac{13}{\varepsilon_0}\|\nabla u\|_2^2\right)$$

with the constants ε_0 and L_0 as in Theorem 2.1.

To complete the proof of Theorem 1.1, we dispense with the assumption that $u \in C_0^\infty(\mathbb{R}^2)$. To this end, assume that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ is such that for its distributional gradient we have $\nabla u \in L^2(\mathbb{R}^2)$. As we already mentioned in Remark 2.9(a), there exists a sequence (u_i) in $C_0^\infty(\mathbb{R}^2)$ such that $u_i \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\nabla u_i \to \nabla u$ in $L^2(\mathbb{R}^2)$.

Let now $f_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a quasiconformal mapping with

$$J_{f_i} \simeq e^{2u_i}, \quad i = 1, 2, \dots, \tag{2.10}$$

where the constants of comparability are independent of i. We may also assume that the mappings are normalized such that $f_i(0) = 0$ for each i. Moreover, each f_i is K-quasiconformal for some $K \ge 1$ depending only on $\|\nabla u\|_2$. We claim that the sequence (f_i) subconverges uniformly on compacta to a K-quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, and that

$$J_f \simeq e^{2u}, \tag{2.11}$$

where the constants of comparability depend only on $\|\nabla u\|_2$.

To this end, we first invoke Trudinger's inequality, [GT, Lemmas 7.13 and 7.16],

$$⨍_B \exp\left(a\frac{|v - v_B|^2}{\|\nabla v\|_2^2}\right) dm_2 \le A, \tag{2.12}$$

valid for all functions $v \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ with $\nabla v \in L^2(\mathbb{R}^2)$. Here $B \subset \mathbb{R}^2$ is an arbitrary ball, and a and A are positive absolute constants. Now by combining (2.12) with the inequality

$$v - v_B \le \frac{4}{a}\|\nabla v\|_2^2 + \frac{a}{4}\frac{|v - v_B|^2}{\|\nabla v\|_2^2},$$

we have that

$$\int_B e^{4v}\, dm_2 \le ACm_2(B)e^{4v_B}, \tag{2.13}$$

where $C \ge 1$ depends on $\|\nabla v\|_2$ only. Now fix an arbitrary ball $B \subset \mathbb{R}^2$. Since $u_i \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$, it follows that $(u_i)_B \to u_B$. In particular, the sequence $((u_i)_B)$ is bounded.

Using this, (2.13) for the functions u and u_i with a uniform constant C, and Bunyakovskiĭ's inequality, we arrive at

$$\begin{aligned}\Big|\int_B (e^{2u_i}-e^{2u})\,dm_2\Big| &\le \int_B |u_i-u|(e^{2u_i}+e^{2u})\,dm_2\\ &\le \left(\left(\int_B e^{4u_i}\,dm_2\right)^{1/2}+\left(\int_B e^{4u}\,dm_2\right)^{1/2}\right)\left(\int_B |u_i-u|^2\,dm_2\right)^{1/2}\\ &\lesssim \left(\int_B |u_i-u|^2\,dm_2\right)^{1/2}\to 0\end{aligned}$$

as $i\to\infty$. We thus conclude that

$$e^{2u_i}\to e^{2u} \tag{2.14}$$

in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

It follows from (2.10), (2.14), and standard convergence results for quasiconformal mappings ([Res, Section II.9], [V, Theorem 20.5]) that the sequence (f_i) subconverges to a (nonconstant) quasiconformal mapping $f\colon\mathbb{R}^2\to\mathbb{R}^2$ locally uniformly. Passing to a subsequence if necessary, we may assume that the original sequence (f_i) converges locally uniformly to f. It is well-known that this implies the weak convergence of the corresponding Jacobians:

$$\int_{\mathbb{R}^2}\varphi J_{f_i}\,dm_2\to\int_{\mathbb{R}^2}\varphi J_f\,dm_2 \tag{2.15}$$

for each $\varphi\in C_0^\infty(\mathbb{R}^2)$; see [Res, Corollary, p. 141, and Theorem 9.1, p. 216]. Thus, by combining (2.10), (2.14), and (2.15), we infer that $e^{2u}\simeq J_f$ as desired. Note that here the constant of comparability depends only on the constant of comparability in (2.10), and hence on $\|\nabla u\|_2$. The same is true for the dilatation of f.

The proof of Theorem 1.1 is complete. □

3 Bessel potentials and strong A_∞-weights.

In this section, we prove the following theorem:

Theorem 3.1 *Let $n\ge 2$, $0<\alpha<n$, and let u belong to the Bessel potential space $L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$. Then*

$$w=e^{nu}$$

is a strong A_∞-weight with data depending only on n, α, and the $L^{\alpha,\frac{n}{\alpha}}$-norm of u.

Recall that, for $\alpha\in\mathbb{R}$ and $1<p<\infty$, the *Bessel potential space* is

$$L^{\alpha,p}(\mathbb{R}^n)=G_\alpha * L^p(\mathbb{R}^n),$$

where G_α is the *Bessel kernel of order* α defined via its Fourier transform

$$\hat{G}(\xi) = \left(1 + |\xi|^2\right)^{-\alpha/2}.$$

By the well-known theorem of Calderón, $L^{\alpha,p}(\mathbb{R}^n)$ coincides with the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ if α is a positive integer. See [AH, Chapter 1] for discussion and references.

Remark 3.2 As remarked in the introduction, we do not know whether functions in $G_\alpha * L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ are comparable to quasiconformal Jacobians in general if $0 < \alpha \leq 1$. Remember however that $G_1 * L^2(\mathbb{R}^2) = W^{1,2}(\mathbb{R}^2)$; so in this case the answer is "yes" by Theorem 1.1.

We begin with a series of lemmas. For the most part, these lemmas are well-known, but partly in lack of a precise reference, and partly for the reader's convenience, we provide detailed proofs. In the following, we shall always assume $n \geq 2$.

Lemma 3.3 *Let $0 < \alpha < n$, and let $p = n/\alpha$. If $f \in L^p(\mathbb{R}^n)$ is supported in the set $\mathbb{R}^n \backslash B(a, 5R_0)$, where $a \in \mathbb{R}^n$ and $R_0 > 0$, then*

$$|G_\alpha * f(x_1) - G_\alpha * f(x_2)| \leq C(n,\alpha)\frac{|x_1 - x_2|}{R_0}\|f\|_p, \tag{3.4}$$

whenever $x_1, x_2 \in B(a, R_0)$.

Proof: For $y \in \mathbb{R}^n \backslash B(a, 5r)$ we have

$$\begin{aligned}|G_\alpha(x_1 - y) - G_\alpha(x_2 - y)| &= \Big|\int_0^1 \nabla G_\alpha(x_1 - y + t(x_2 - x_1)) \cdot (x_2 - x_1)\, dt\Big| \\ &\leq C(n,\alpha)\frac{|x_1 - x_2|}{|y - a|^{n+1-\alpha}},\end{aligned}$$

where the inequality follows from the properties of the Bessel kernel [AH, pp. 12–13]. Therefore,

$$\begin{aligned}&|G_\alpha * f(x_1) - G_\alpha * f(x_2)| \\ \leq\ &\int_{\mathbb{R}^n \backslash B(a,5R_0)} |G_\alpha(x_1 - y) - G_\alpha(x_2 - y)|\, |f(y)|\, dm_n(y) \\ \leq\ &C|x_1 - x_2| \int_{\mathbb{R}^n \backslash B(a,5R_0)} \frac{|f(y)|}{|y - a|^{n+1-\alpha}}\, dm_n(y) \\ \leq\ &C\frac{|x_1 - x_2|}{R_0}\|f\|_p,\end{aligned}$$

where Hölder's inequality was used in the last step. The lemma follows. □

We let $\mathcal{H}^1_\infty$ denote the Hausdorff 1-content defined as

$$\mathcal{H}^1_\infty(E) = \inf\left\{\sum_i \operatorname{diam}(B_i) : E \subset \bigcup_i B_i\right\}$$

for $E \subset \mathbb{R}^n$. Here the infimum is taken over all countable covers (B_i) of E by open balls.

Lemma 3.5 (Cartan's lemma) *Let μ be a finite positive measure on $\mathbb{R}^n$. Then for each $\varepsilon > 0$ there exists a set $E \subset \mathbb{R}^n$ such that*

(i) $\mathcal{H}^1_\infty(E) \leq \varepsilon$ *and that*

(ii) $\mu(B(x,r)) \leq \frac{10r}{\varepsilon}\mu(\mathbb{R}^n)$, *whenever $x \in \mathbb{R}^n \backslash E$ and $r > 0$.*

Proof: We may assume $\mu(\mathbb{R}^n) > 0$. Fix $\varepsilon > 0$, and let E denote the set of the points x in $\mathbb{R}^n$ for which the inequality in (ii) fails for some $r > 0$. Thus, for each $x \in E$ there exists $r_x > 0$ such that

$$\mu(B(x,r_x)) > \frac{10r_x}{\varepsilon}\mu(\mathbb{R}^n).$$

We note that the supremum of all such radii r_x does not exceed $\varepsilon/10$. This allows us to apply a standard covering lemma [H, p. 2] and select a countable sequence of points (x_i) in E with the property that the corresponding balls $B(x_i, r_{x_i})$ are pairwise disjoint and that

$$E \subset \bigcup_i B(x_i, 5r_{x_i}).$$

Therefore,

$$\mathcal{H}^1_\infty(E) \leq 10\sum_i r_{x_i} \leq \frac{\varepsilon}{\mu(\mathbb{R}^n)}\sum_i \mu(B(x_i, r_{x_i})) \leq \varepsilon$$

as desired. The lemma follows. □

Lemma 3.6 *Let $0 < \alpha < n$, and let $p = n/\alpha$. If $f \in L^p(\mathbb{R}^n)$ is supported in $B(a, R_0)$, where $a \in \mathbb{R}^n$ and $R_0 > 0$, then for each $\varepsilon > 0$ we have that*

$$\mathcal{H}^1_\infty\left(\left\{x \in B(a,R_0) : |G_\alpha * f(x)| > c_0\|f\|_p(1/\varepsilon)^{1/p}\right\}\right) \leq \varepsilon R_0, \tag{3.7}$$

where $c_0 > 0$ depends only on n and α.

Proof: Fix $\varepsilon > 0$. We apply Lemma 3.5 with $d\mu = |f|^p\, dm_n$, and find a set $E \subset \mathbb{R}^n$ such that

$$\mathcal{H}^1_\infty(E) \leq \varepsilon R_0 \tag{3.8}$$

and that

$$\int_{B(x,r)} |f|^p\, dm_n \leq \frac{10r}{\varepsilon R_0}\int_{\mathbb{R}^n} |f|^p\, dm_n, \tag{3.9}$$

whenever $x \in \mathbb{R}^n \backslash E$ and $r > 0$.

Now fix $x \in (\mathbb{R}^n \backslash E) \cap B(a, R_0)$. We have that

$$|G_\alpha * f(x)| \le C(n,\alpha) \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dm_n(y) \tag{3.10}$$

by the properties of the Bessel kernel [AH, (1.2.12)]. Upon writing $B_\nu = B(x, 2^{1-\nu}R_0)$ for $\nu = 0, 1, \ldots,$ we estimate

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dm_n(y) &= \sum_{\nu=0}^{\infty} \int_{B_\nu \backslash B_{\nu+1}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dm_n(y) \\ &\le CR_0^{\alpha-n} \sum_{\nu=0}^{\infty} 2^{(n-\alpha)\nu} \Big(\int_{B_\nu} |f|^p\, dm_n \Big)^{1/p} \Big(2^{-\nu n} R_0^n\Big)^{\frac{n-\alpha}{n}} \\ &= C \sum_{\nu=0}^{\infty} \Big(\int_{B_\nu} |f|^p\, dm_n \Big)^{1/p} \\ &\le C\|f\|_p \sum_{\nu=0}^{\infty} \left(\frac{2^{-\nu}}{\varepsilon} \right)^{1/p} = C\|f\|_p (1/\varepsilon)^{1/p}, \end{aligned}$$

where $C = C(n,\alpha) > 0$ and where (3.9) was used in the last inequality. By combining this with (3.10) and (3.8), we find that (3.7) holds provided $c_0 = c_0(n,\alpha) > 0$ is chosen large enough. The lemma follows. □

Lemma 3.11 *Let $x, y \in \mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a Borel set. Suppose that $B_1, \ldots, B_k$ are open balls in $\mathbb{R}^n$ such that $x \in B_1$, $y \in B_k$, and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$. Then there exists a constant $c_1 = c_1(n) > 0$ with the following property: if*

$$\mathcal{H}^1_\infty(E) \le c_1 |x-y|, \tag{3.12}$$

then

$$\sum_{i \in \mathcal{G}} \operatorname{diam}(B_i) > \frac{1}{10}|x-y|, \tag{3.13}$$

where

$$\mathcal{G} = \left\{ i = 1, \ldots, k : m_n(E \cap B_i) \le \frac{1}{2} m_n(B_i) \right\}. \tag{3.14}$$

Proof: Let L be the 1-dimensional affine line passing through x and y. For notational simplicity we identify L with the first coordinate axis $\mathbb{R} \subset \mathbb{R}^n$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}$ be the projection onto $\mathbb{R}$, and denote $A' = \pi(A)$ for $A \subset \mathbb{R}^n$. Note that A' is Lebesgue measurable, whenever A is a Borel set.

It follows from Fubini's theorem that there exists $\delta_0 = \delta_0(n) > 0$ such that

$$m_n(E \cap B) \le \frac{1}{2} m_n(B),$$

whenever B is a ball in $\mathbb{R}^n$ with

$$m_1(E' \cap B') \le \delta_0 m_1(B').$$

If we now let

$$\mathcal{G}_0 = \{i = 1, \dots, k : m_1(E' \cap B_i') \le \delta_0 m_1(B_i')\},$$

we observe that $\mathcal{G}_0 \subset \mathcal{G}$, and hence it is enough to show that (3.13) holds for $\mathcal{G}_0$ in place of $\mathcal{G}$.

To this end, let $\mathcal{F} \subset \{1, \dots, k\}$ be such that the intervals B_i', $i \in \mathcal{F}$, are pairwise disjoint and

$$[x, y] \subset B_1' \cup \dots \cup B_k' \subset \bigcup_{i \in \mathcal{F}} 5B_i' \,,$$

where $5B_i$ denotes the interval with the same center as B_i' but 5 times longer (cf. [H, p. 2]). Note that for a measurable subset $M \subset \mathbb{R}$ we have $\mathcal{H}^1_\infty(M) = m_1(M)$. Hence

$$\begin{aligned}
\mathcal{H}^1_\infty(E) &\ge \mathcal{H}^1_\infty(E') = m_1(E') \\
&\ge \sum_{i \in \mathcal{F}} m_1(E' \cap B_i') \ge \delta_0 \sum_{i \in \mathcal{F} \setminus \mathcal{G}_0} m_1(B_i') \\
&\ge \delta_0 \Big(\sum_{i \in \mathcal{F}} m_1(B_i') - \sum_{i \in \mathcal{G}_0} m_1(B_i') \Big) \\
&\ge \frac{\delta_0}{5} |x - y| - \delta_0 \sum_{i \in \mathcal{G}_0} \operatorname{diam}(B_i).
\end{aligned}$$

By choosing $c_1 = c_1(n) > 0$ in (3.12) small enough, we arrive at the desired conclusion. The lemma follows. $\square$

Before we begin the proof of Theorem 3.1 in earnest, we recall the following inequality, valid for each f in $L^p(\mathbb{R}^n)$ supported in a ball $B(a, R_0)$, where $a \in \mathbb{R}^n$ and $R_0 > 0$:

$$\int_{B(a,R_0)} \exp\left(\frac{\beta |(G_\alpha * f)(x)|^{p'}}{\|f\|_p^{p'}} \right) dm_n(x) \le C R_0^n, \tag{3.15}$$

with $0 < \alpha < n$, $p = n/\alpha$, $p' = \frac{n}{n-\alpha}$, and with constants $\beta = \beta(n, \alpha) > 0$ and $C = C(n, \alpha) \ge 1$. Inequality (3.15) follows from [AH, (1.2.12) and Theorem 3.1.4, p. 56].

Proof of Theorem 3.1: Let $0 < \alpha < n$, $p = n/\alpha$, and let $u = G_\alpha * f$ for some $f \in L^p(\mathbb{R}^n)$. In the following, C will denote a generic positive constant that depends only on α, n, and the L^p-norm of f.

We write $d\mu(x) = w(x)\, dm_n(x)$, where $w(x) = e^{nu(x)}$. Recall the notation $d_\mu(x, y)$ and B_{xy} from (1.13). We shall show both that μ is a doubling measure and that

$$\delta_\mu(x, y) := \inf \sum_{i=1}^k \mu(B_i)^{1/n} \ge \frac{1}{C} \mu(B_{xy})^{1/n} = \frac{1}{C} d_\mu(x, y) \tag{3.16}$$

for all $x, y \in \mathbb{R}^n$, where the infimum is taken over finite chains of open balls connecting x and y; that is,

$$x \in B_1,\ y \in B_k \ \text{ and } \ B_i \cap B_{i+1} \neq \emptyset \ \text{ for all } \ i = 1, \ldots, k-1. \tag{3.17}$$

Indeed, (3.16) implies both that δ_μ is a metric and that it is comparable to d_μ as required in (1.15). (It is easy to see that a doubling measure μ is a metric doubling measure if and only if (3.16) holds.)

Towards this end, fix $x, y \in \mathbb{R}^n$ and let

$$a = \frac{x+y}{2}, \quad R = |x-y|, \quad B = B(a,R), \quad \lambda B = B(a, \lambda R) \ \text{ for } \ \lambda > 0.$$

Let us write $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & x \in 10B, \\ 0, & x \in \mathbb{R}^n \setminus 10B, \end{cases}$$

and $f_2 = f - f_1$. Put

$$u_i = G_\alpha * f_i \ , \qquad i = 1, 2,$$

so that $u = u_1 + u_2$. We claim that

$$d_\mu(x,y) \le \mu(2B)^{1/n} \le CR \exp(u_2(a)). \tag{3.18}$$

The first inequality is clear. To see the second, apply Lemma 3.3 to conclude that

$$|u_2(z) - u_2(a)| \le C \quad \text{for} \quad z \in 2B. \tag{3.19}$$

Now assume that $\|f_1\|_p > 0$. If we use Young's inequality

$$st \le \varepsilon s^p + \varepsilon^{-1/(p-1)}\ t^{p'}, \quad s,t \ge 0,\ \varepsilon > 0,\ p' = \frac{p}{p-1},$$

for $s = n\|f_1\|_p$, $t = u_1(z)/\|f_1\|_p$ and $\varepsilon = \beta^{-(p-1)}$, where $\beta > 0$ is the constant in (3.15), then we get

$$n u_1(z) \le \beta^{-(p-1)} n^p \|f\|_p^p + \beta \frac{u_1(z)^{p'}}{\|f_1\|_p^{p'}}. \tag{3.20}$$

By combining (3.19), (3.20), and (3.15), we find that

$$\begin{aligned} \mu(2B) &= \int_{2B} w(z)\, dm_n(z) = \int_{2B} \exp\Big(n(u_1(z) + u_2(z))\Big)\, dm_n(z) \\ &\le C \exp(n u_2(a)) \int_{10B} \exp(n u_1(z))\, dm_n(z) \\ &\le C \exp(n u_2(a)) \int_{10B} \exp\left(\beta \frac{u_1(z)^{p'}}{\|f_1\|_p^{p'}}\right) dm_n(z) \\ &\le C \exp(n u_2(a)) R^n. \end{aligned}$$

This proves (3.18) if $\|f_1\|_p > 0$. If $\|f_1\|_p = 0$, then $u_1 = 0$, and (3.18) follows directly from (3.19).

Next, choose $f = f_1$, $R_0 = 10R$, and $\varepsilon = c_1/10$ in Lemma 3.6, where $c_1 = c_1(n) > 0$ is the constant from Lemma 3.11. It follows that there is a Borel set $E \subset 10B$ such that

$$\mathcal{H}^1_\infty(E) \leq c_1 R = c_1|x-y| \tag{3.21}$$

and

$$|u_1(z)| \leq C \quad \text{for all} \quad z \in 10B \backslash E. \tag{3.22}$$

Let $B_1, \dots, B_k$ be an arbitrary chain of balls connecting x and y as in (3.17). Then we obtain from (3.21) and from Lemma 3.11 that

$$\sum_{i \in \mathcal{G}} \operatorname{diam}(B_i) > \frac{1}{10} R, \tag{3.23}$$

where $\mathcal{G}$ is given in (3.14).

Now assume that $B_i \subset 2B$ for all $i = 1, \dots, k$. Then it follows from (3.19), (3.22), and (3.23) that

$$\begin{aligned}
\sum_{i=1}^{k} \Big(\int_{B_i} w(z)\, dm_n(z) \Big)^{1/n} &\geq \sum_{i \in \mathcal{G}} \Big(\int_{B_i \backslash E} w(z)\, dm_n(z) \Big)^{1/n} \\
&\geq \frac{1}{C} \sum_{i \in \mathcal{G}} \Big(\int_{B_i \backslash E} \exp(n u_2(a))\, dm_n(z) \Big)^{1/n} \\
&= \frac{1}{C} \exp(u_2(a)) \sum_{i \in \mathcal{G}} m_n(B_i \backslash E)^{1/n} \\
&\geq \frac{1}{C} \exp(u_2(a)) \sum_{i \in \mathcal{G}} \operatorname{diam}(B_i) \\
&\geq \frac{1}{C} R \exp(u_2(a)).
\end{aligned} \tag{3.24}$$

Using (3.24) for the chain consisting of the single ball $B_1 = B \supset \{x, y\}$, we see that

$$R^n \exp(n u_2(a)) \leq C \mu(B). \tag{3.25}$$

Hence by (3.18) we have that

$$\mu(2B) \leq C \mu(B).$$

Here B is essentially an arbitrary ball, since x and y were arbitrary. So the last inequality implies that μ is a doubling measure. Recall that the constant C depends only on n, α, and $\|f\|_p$, and in particular not on B.

Next, if the chain (B_i) does not lie entirely in $2B$, then there exists a smallest number k' with $1 \leq k' \leq k$ such that $B_{k'}$ contains a point y' with $|x-y'| = R$. Then $B_1, \ldots, B_{k'}$ is a chain of balls connecting x and y'. Note that the definition of k' implies

$$B_1 \cup \ldots \cup B_{k'-1} \subset B(x,R) \subset \tfrac{3}{2}B.$$

If $B_{k'} \subset 2B$, then the subchain $B_1, \ldots, B_{k'}$ is contained in $2B$ and we can apply the preceding argument with y' in place of y to conclude that (3.24) holds; in the opposite case, $\operatorname{diam}(B_{k'}) \geq R/2$. The doubling condition for μ and (3.25) then imply

$$R^n \exp(nu_2(a)) \leq C\mu(B) \leq C\mu(B_{k'}),$$

and so again (3.24) is true.

Thus, (3.24) is true in all cases, and inequality (3.16) follows from combining (3.24) and (3.18). This completes the proof of Theorem 3.1. □

Remark 3.26 The conclusion in Theorem 3.1 remains valid for *Riesz potentials*

$$u(x) = (I_\alpha * g)(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dm_n(y), \tag{3.27}$$

where $0 < \alpha < n$, and $g \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ is compactly supported. We have to assume that g is compactly supported, because otherwise the integral in (3.27) might not exist. The proof of Theorem 3.1 for Riesz potentials is essentially the same as in the Bessel potential case. To see this note that the only properties of Bessel potentials we used were the inequalities in (3.15), Lemma 3.3 and Lemma 3.6. These statements are equally valid for Riesz potentials of compactly supported functions.

4 Concluding remarks and questions.

Let us first discuss Theorem 1.1. A function $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ with L^2-integrable distributional gradient can attain values $\pm\infty$ only on a set of 2-capacity zero. Thus the Jacobian determinant J_f of the mapping f given by Theorem 1.1 can only degenerate (assume the value 0) or blow up (assume the value ∞) on a relatively small set. On the other hand, for each set $E \subset \mathbb{R}^2$ of zero 2-capacity there exists a function $u \in W^{1,2}(\mathbb{R}^2)$ such that

$$\lim_{z \to x} u(z) = \infty, \tag{4.1}$$

whenever $x \in E$. Naturally, functions in $W^{1,2}(\mathbb{R}^2)$ need not be continuous and their singularities as in (4.1) must be understood properly. One way is to consider *quasicontinuous* representations of functions in $W^{1,2}(\mathbb{R}^2)$, or else limits

$$\lim_{r \to 0} \fint_{B(x,r)} u(z)\, dm_2(z)\ , \tag{4.2}$$

which exist at 2-capacity quasievery point $x \in \mathbb{R}^2$. See [MZ, Section 2.1].

In any case, it follows that *given a set* $E \subset \mathbb{R}^2$ *of 2-capacity zero, there exists* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *quasiconformal such that*

$$\lim_{r \to 0} \fint_{B(x,r)} J_f(z)\, dm_n(z) = \infty \tag{4.3}$$

for each $x \in E$. Alternatively, one could prescribe the limit in (4.3) to be 0. Although sets of 2-capacity zero can be uncountable and dense in $\mathbb{R}^2$ for example, they are still metrically small, since they have Hausdorff dimension zero.

One would expect that sets of singularities, say in the sense of (4.3), can be much larger than of Hausdorff dimension zero. This is indeed the case for the map g given in (1.11).

Question 4.4 *If $E \subset \mathbb{R}^2$ is a Borel set of Lebesgue 2-measure zero, is it then true that there exists a quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that (4.3) holds for each $x \in E$?*

One can ask a similar question, where the limit in (4.3) is required to be 0.

The answer to Question 4.4 is "yes" if the answer to the following very interesting question is "yes":

Question 4.5 (See [HS, Question 2]) *Is every A_1-weight in $\mathbb{R}^2$ comparable to a quasiconformal Jacobian?*

Recall that a weight w in $\mathbb{R}^n$ is an A_1-weight if there exists a constant $C \geq 1$ such that

$$\fint_B w(x)\, dm_n(x) \leq C \operatorname{essinf}_B w$$

for each ball $B \subset \mathbb{R}^n$. One can show that for each Borel set $E \subset \mathbb{R}^n$ of Lebesgue n-measure zero there exists an A_1-weight w such that $w(z) \to \infty$ as $z \to x$ for $x \in E$.

Naturally, questions analogous to questions (4.4) and (4.5) can be asked in every dimension $n \geq 2$.

Here is the next question:

Question 4.6 *Are the weights of the form*

$$w = e^{nu}, \qquad u \in L^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n), \quad 0 < \alpha < n, \tag{4.7}$$

as in Theorem 3.1 comparable to quasiconformal Jacobians for each $n \geq 2$?

Recall from the introduction that an affirmative answer to question (4.6) is equivalent to the requirement that the (quasi)metric space $(\mathbb{R}^n, d_\mu)$, where $d\mu = w\, dm_n$, is bi-Lipschitz equivalent to $\mathbb{R}^n$. We can combine Theorem 3.1 with a result of Semmes [S1, Theorem 5.2] to obtain the following theorem:

Theorem 4.8 *Suppose w is a weight in $\mathbb{R}^n$, $n \geq 2$ satisfying (4.7). Then there is a bi-Lipschitz embedding of the (quasi)metric space $(\mathbb{R}^n, d_\mu)$ into some finite-dimensional Euclidean space.*

Indeed, Semmes showed in [S1] that if a strong A_∞-weight w in $\mathbb{R}^n$ has the stability property that $w^{1\pm\varepsilon}$ is a strong A_∞-weight for all sufficiently small $\varepsilon > 0$, then the conclusion of Theorem 4.8 holds (with $d\mu = w\, dm_n$). Semmes called weights with this stability property *stronger A_∞-weights*. Obviously, weights as in (4.7) are stronger A_∞-weights, whence the preceding theorem.

The fact that the weights as in (4.7) are stronger A_∞-weights is obvious from the definition. We can express this fact by an inclusion

$$L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n) \subset \log(A_\infty^s(\mathbb{R}^n)), \quad 0 < \alpha < n, \tag{4.9}$$

where

$$A_\infty^s(\mathbb{R}^n) := \{w : w \text{ is a stronger } A_\infty\text{-weights in } \mathbb{R}^n\}.$$

If we define

$$\mathcal{J}(\mathbb{R}^n) = \{J_f : f\colon \mathbb{R}^n \to \mathbb{R}^n \text{ quasiconformal}\},$$

then Theorem 1.1 implies the inclusion

$$W^{1,2}(\mathbb{R}^2) \subset \log(\mathcal{J}(\mathbb{R}^2)) + L^\infty(\mathbb{R}^2). \tag{4.10}$$

The set $\log(\mathcal{J}(\mathbb{R}^n)) + L^\infty(\mathbb{R}^n)$ is a subset of $\mathrm{BMO}(\mathbb{R}^n)$ as discussed in the introduction, and the quasiconformal Jacobian problem seeks to understand this set. What we have found in (4.10) is that this set (for $n = 2$) contains the infinite dimensional subspace $W^{1,2}(\mathbb{R}^2)$ of $\mathrm{BMO}(\mathbb{R}^n)$.

Similarly, $\log(A_\infty^s(\mathbb{R}^n))$ is a subset of $\mathrm{BMO}(\mathbb{R}^n)$, strictly larger than $\log(\mathcal{J}(\mathbb{R}^n)) + L^\infty(\mathbb{R}^n)$ by work of Semmes and of Laakso as explained in the introduction. Inclusion (4.9) implies that this set contains the infinite dimensional subspace $L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ of $\mathrm{BMO}(\mathbb{R}^n)$. (The fact that $L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ is rather well-known [AH, p. 79].)

Question 4.11 *Are there other linear subspaces of* $\mathrm{BMO}(\mathbb{R}^n)$ *that are contained in the sets* $\log(\mathcal{J}(\mathbb{R}^n)) + L^\infty(\mathbb{R}^n)$ *and* $\log(A_\infty^s(\mathbb{R}^n))$*?*

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