# RIGIDITY FOR QUASI-FUCHSIAN ACTIONS ON NEGATIVELY CURVED SPACES

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Abstract. We prove that if $G \curvearrowright X$ is a convex cocompact isometric group action on a CAT$(-1)$ space, and the limit set has Hausdorff and topological dimensions equal to 1, then the action preserves a convex subset $Y \subseteq X$ isometric to $\mathbb{H}^2$. This implies a conjecture by M. Bourdon. Our methods also yield a new proof of the $n$-dimensional analog of this statement for $n \geq 2$.

## 1. Introduction

The purpose of this note is to prove the following theorem.

**Theorem 1.1.** *Let $G \curvearrowright X$ be a properly discontinuous, quasi-convex cocompact, and isometric action of a group $G$ on a* CAT$(-1)$*-space $X$. If the Hausdorff dimension and topological dimension of the limit set $\Lambda(G) \subseteq \partial_\infty X$ are both equal to 1, then $X$ contains a convex and $G$-invariant set $Y$ isometric to the hyperbolic plane $\mathbb{H}^2$ on which $G$ acts cocompactly.*

A similar statement is true if we assume that the Hausdorff dimension and the topologial dimension of the limit set $\Lambda(G)$ are both equal to some integer $n \geq 2$; this was shown by the authors in [1] where the case $n = 1$ was left open. Our proof depended on a rigidity result by M. Bourdon [2, 0.3 Théorème, ($\mathbb{H}^n$ case)] which made the assumption $n \geq 2$ necessary. Here we employ a different line of reasoning, using a more elementary fact [2, 0.1 Théorème] (cf. Proposition 2.4). In fact this argument can be adapted to give a new geometric proof of [2, 0.3 Théorème, ($\mathbb{H}^n$ case)] which avoids the analytic considerations in [2, Section 2] (see Section 5).

In his paper Bourdon conjectured that if $G$ is a cocompact Fuchsian group that acts on a CAT$(-1)$-space $X$ as in Theorem 1.1 and if the Hausdorff dimension of $\Lambda(G)$ is equal to 1, then $G$ stabilizes a convex copy of $\mathbb{H}^2$ in $X$ on which $G$ acts cocompactly. In this situation the

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limit set $\Lambda(G)$ is a topological circle, so his conjecture is implied by our more general statement.

The proof of Theorem 1.1 is based on the following observation, which may have further applications. If $G \curvearrowright X$ is a group action as in the theorem (without the dimensional restriction), then every weak tangent of the limit set $\Lambda(G)$ is *isometric* to the set $\Lambda(G)$ minus a point equipped with a *parabolic visual metric* (see Proposition 3.1). This parabolic metric can be obtained as a suitably normalized limit of standard visual metrics if we let the basepoints of the metrics tend to the boundary. A similar fact turned out to be useful in the context of quasi-Möbius actions (cf. [1, Lemma 5.3]) with the essential difference that it was possible to identify the weak tangents only up to quasisymmetric equivalence instead of isometry.

## 2. CAT(−1)-SPACES AND ISOMETRIC GROUP ACTIONS

We briefly summarize some facts about CAT(−1)-spaces. Our terminology is standard, and we assume that the reader is familiar with the basic definitions. For a more detailed discussion see [2].

A CAT(−1)-space $X$ is a proper geodesic metric space whose geodesic triangles are in an appropriate sense "thinner" than comparison triangles in the model space $\mathbb{H}^2$. We denote the distance between two points $x$ and $y$ in $X$ by $d(x, y)$. The space $X$ can be compactified by adding a boundary at infinity $\partial_\infty X$ which is defined by using asymptotic classes of geodesic rays. If $p, q \in X$, then we denote by $\overline{pq}$ the geodesic segment with endpoints $p$ and $q$. Similar notation will be used for geodesic rays and lines.

For $\xi \in \partial_\infty X$ and $x, y \in X$ the *Busemann function* is defined as

$$B_\xi(x, y) := \lim_{t\to\infty} (d(x, r(t)) - d(y, r(t))).$$

Here $t \mapsto r(t)$ is a parametrized geodesic ray asymptotic to $\xi$. One can show that the limit exists and is independent of the choice of the geodesic ray representing $\xi$. Note that the level sets of the function $B_\xi(x, \cdot)$ are the horospheres centered at $\xi$, normalized so that the level 0 corresponds to the horosphere passing through $x$.

For a basepoint $p \in X$ and points $\xi, \eta \in \partial_\infty X$, $\xi \neq \eta$, one defines the *Gromov product* as

$$(\xi, \eta)_p := \tfrac{1}{2}(B_\xi(p, z) + B_\eta(p, z)).$$

Here $z$ is an arbitrary point on the geodesic connecting $\xi$ and $\eta$. The definition is independent of the choice of $z$, since $2(\xi, \eta)_p$ is equal to the length of the geodesic segment that is cut out from $\overline{\xi\eta}$ by the intersection of the horoballs $\{u \in X : B_\xi(p,u) \geq 0\}$ and $\{u \in X : B_\eta(p,u) \geq 0\}$. One can show that if $\xi, \eta \in \partial_\infty X$ and one defines

$$d_p(\xi,\eta) := e^{-(\xi,\eta)_p},$$

for $\xi \neq \eta$ and $d_p(\xi,\eta) = 0$ for $\xi = \eta$, then $d_p$ is a metric on $\partial_\infty X$. This metric is called *visual metric with basepoint* $p$.

A "parabolic" analog of the visual metrics $d_p$ can be obtained as follows. Fix $\zeta \in \partial_\infty X$ and $q \in X$. Let $\xi$ and $\eta$ be distinct points in $\partial_\infty X \setminus \{\zeta\}$. Then there exist unique points $x \in \overline{\xi\zeta}$ and $y \in \overline{\eta\zeta}$ such that $B_\xi(q,x) = B_\eta(q,y) = 0$. Now if $z$ is an arbitrary point on $\overline{\xi\eta}$, we let

$$(\xi,\eta)_{\zeta,q} := \tfrac{1}{2}(B_\xi(x,z) + B_\eta(y,z)).$$

Again one can show that this quantity is independent of $z$. We now define

$$d_{\zeta,q}(\xi,\eta) := e^{-(\xi,\eta)_{\zeta,q}}.$$

If $\xi = \eta$ we let $d_{\zeta,q}(\xi,\eta) := 0$.

**Lemma 2.1.** *Suppose $X$ is a* CAT$(-1)$*-space, $q \in X$, $\zeta \in \partial_\infty X$, and $\xi, \eta \in \partial_\infty X \setminus \{\zeta\}$, $\xi \neq \eta$. Then,*

(i) $(\xi,\eta)_{\zeta,q} = \lim_{p\to\zeta} \big((\xi,\eta)_p - d(p,q)\big)$,

(ii) *$d_{\zeta,q}$ is a metric on $\partial_\infty X \setminus \{\zeta\}$.*

*Proof.* For $p \in X$ we define $(\xi,\eta)_{p,q}$ in a similar way as $(\xi,\eta)_{\zeta,q}$. Namely, there exist points $x' \in \overline{p\xi}$ and $y' \in \overline{p\eta}$ such that $d(x',p) = d(y',p) = d(q,p)$. Now if $z \in \overline{\xi\eta}$ is arbitrary, then we let

$$(\xi,\eta)_{p,q} := \tfrac{1}{2}(B_\xi(x',z) + B_\eta(y',z)).$$

Again this quantity is independent of $z$. In fact

$$(\xi,\eta)_{p,q} = (\xi,\eta)_p - d(q,p). \tag{2.2}$$

On the other hand, if $p$ tends to $\zeta$, then the sphere centered at $p$ passing through $q$ "tends" to the horosphere centered at $\zeta$ passing through $q$. Moreover, the geodesic rays $\overline{p\xi}$ and $\overline{p\eta}$ "tend" to $\overline{\zeta\xi}$ and $\overline{\zeta\eta}$, respectively. This implies that the auxiliary points $x'$ and $y'$ used in the definition of $(\xi,\eta)_{p,q}$ tend to the points $x$ and $y$, respectively, that were used in the definition of $(\xi,\eta)_{\zeta,q}$. So

$$\lim_{p\to\zeta} (\xi,\eta)_{p,q} = (\xi,\eta)_{\zeta,q},$$

and we get (i).

The second statement (ii) follows from (i), the fact that $d_p$ for $p \in X$ is a metric, and a limiting argument. ☐

It is not hard to see that if we represent $\mathbb{H}^n$, $n \geq 2$, by the upper-half space model so that $\partial_\infty \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$, then $d_{\infty,q}$ for $q \in \mathbb{H}^n$ is just a rescaling of the Euclidean metric on $\mathbb{R}^{n-1}$.

For four distinct points $\xi_1, \xi_2, \xi_3, \xi_4 \in \partial_\infty X$, one can define the *cross-ratio* as

$$[\xi_1, \xi_2, \xi_3, \xi_4] := \frac{d_p(\xi_1, \xi_3) d_p(\xi_2, \xi_4)}{d_p(\xi_1, \xi_4) d_p(\xi_2, \xi_3)}.$$

Here $p$ is an arbitrary point in $X$. The definition is independent of $p$. It turns out that we can also use parabolic metrics in this expression for the cross-ratio.

**Lemma 2.3.** *Suppose $X$ is a* CAT$(-1)$*-space, $q \in X$, $\zeta \in \partial_\infty X$, and $\xi_1, \xi_2, \xi_3, \xi_4 \in \partial_\infty X \setminus \{\zeta\}$ are four distinct points. Then*

$$[\xi_1, \xi_2, \xi_3, \xi_4] = \frac{d_{\zeta,q}(\xi_1, \xi_3) d_{\zeta,q}(\xi_2, \xi_4)}{d_{\zeta,q}(\xi_1, \xi_4) d_{\zeta,q}(\xi_2, \xi_3)}.$$

*Proof.* This is an immediate consequence of the definitions of the cross-ratio and the metric $d_{\zeta,q}$, Lemma 2.1(i), and a limiting argument. □

Suppose $G \curvearrowright X$ is an isometric action of a group on a CAT$(-1)$-space $X$. We denote the limit set of $G$ by $\Lambda(G) \subseteq \partial_\infty X$. A subset $Y \subseteq X$ is *quasi-convex* if there is a constant $C$ such that any geodesic segment with endpoints in $Y$ lies in the $C$-neighborhood of $Y$. The action $G \curvearrowright X$ is *quasi-convex cocompact* if there is a $G$-invariant quasi-convex subset $Y \subseteq X$ on which $G$ acts with compact quotient $Y/G$. The group $G$ is quasi-convex cocompact if and only if all orbits of points in $X$ under $G$ are quasi-convex.

An injective map between metric spaces is called *Möbius* if it preserves cross-ratios. The following proposition was proved in [2, 3.3 Lemme].

**Proposition 2.4.** *Let $f$ be a Möbius embedding of $\mathbb{S}^1 = \partial_\infty \mathbb{H}^2$ into the boundary $\partial_\infty X$ of a* CAT$(-1)$*-space $X$. Then $f$ is the boundary map of an isometry of $\mathbb{H}^2$ onto a totally geodesic subspace of $X$.*

Here it is understood that $\mathbb{S}^1$ and $\partial_\infty X$ are equipped with visual metrics.

## 3. WEAK TANGENTS

In this section we prove our main observation about weak tangents of limit sets on boundaries of CAT$(-1)$-spaces. A complete metric space $(S, \delta)$ is called a *weak tangent* of the metric space $(Z, d)$, if there exist a sequence of numbers $\lambda_k > 0$ with $\lambda_k \to \infty$ for $k \to \infty$ and points $q \in S$, $p_k \in Z$ such that the sequence of pointed metric spaces $(Z, p_k, \lambda_k d)$ converges in the Gromov-Hausdorff topology to the pointed space $(S, q, \delta)$. We assume the reader is familar with these concepts. For more details see [1, Section 4], and for additional background [4] and [3].

**Proposition 3.1.** *Let $G \curvearrowright X$ be a properly discontinuous, quasi-convex cocompact, and isometric action on a* CAT$(-1)$*-space $X$. Then for every weak tangent $S$ of $\Lambda(G)$ equipped with the restriction of a visual metric $d_p$, $p \in X$, there exist points $\zeta \in \Lambda(G)$ and $q \in X$ such that $S$ is isometric to $\Lambda(G) \setminus \{\zeta\}$ equipped with the restriction of the parabolic metric $d_{\zeta,q}$.*

*Proof.* The main point is that a rescaling of a visual metric can be obtained from a geometric construction that is very similar to the definition of the parabolic metric and was described in the proof of Lemma 2.1. More precisely, let $p \in X$ and $\xi^0 \in \partial_\infty X$ be basepoints, and fix $q \in \overline{p\xi^0}$. For distinct points $\xi, \eta \in \partial_\infty X$ define $(\xi, \eta)_{p,q}$ as in the proof of Lemma 2.1. Setting

$$d_{p,q}(\xi, \eta) := e^{-(\xi,\eta)_{p,q}},$$

we obtain a metric $d_{p,q}$ on $\partial_\infty X$. From (2.2) it follows that $d_{p,q} = e^{d(q,p)} d_p$. So the metric $d_{p,q}$ is just a rescaling of $d_p$. By choosing the point $q$ appropriately, we can obtain arbitrary rescaling factors $\geq 1$.

Now suppose $S$ is a weak tangent of $\Lambda(G)$ equipped with the (restriction of the) visual metric $d_p$. By what we have just seen, there exist basepoints $\xi^0_k$ in $\Lambda(G)$ and points $q_k \in \overline{p\xi^0_k}$ for $k \in \mathbb{N}$ such that $d(q_k, p) \to \infty$ and $(\Lambda(G), \xi^0_k, d_{p,q_k}) \to S$ as $k \to \infty$. Since $G \curvearrowright X$ is quasi-convex cocompact, we can find elements $g_k \in G$ for $k \in \mathbb{N}$ so that the sequence $\{g_k(q_k)\}$ stays in a compact subset of $X$. Since $G \curvearrowright X$ is isometric, we have $d(g_k(q_k), g_k(p)) = d(q_k, p) \to \infty$ as $k \to \infty$. Hence $g_k(p) \to \Lambda(G)$ as $k \to \infty$. By passing to appropriate subsequences if necessary, we may without loss of generality assume that the sequences $\{g_k(\xi^0_k)\}$, $\{g_k(q_k)\}$ and $\{g_k(p)\}$ converge; say $\{g_k(\xi^0_k)\} \to \xi^0 \in \Lambda(G)$, $\{g_k(q_k)\} \to q \in X$, and $\{g_k(p)\} \to \zeta \in \Lambda(G)$. Note that then $\xi^0 \neq \zeta$. Since the group $G$ acts by isometries on $X$ and $\Lambda(G)$ is invariant under the group action, the space $S_k := \big(\Lambda(G), g_k(\xi^0_k), d_{g_k(p), g_k(q_k)}\big)$ is isometric to $(\Lambda(G), \xi^0_k, d_{p,q_k})$ by a basepoint preserving isometry for $k \in \mathbb{N}$. Hence $S_k \to S$ as $k \to \infty$. On the other hand, it follows from Lemma 2.1(i) that $S_k \to (\Lambda(G) \setminus \{\zeta\}, \xi^0, d_{\zeta,q})$ as $k \to \infty$. The claim follows. ☐

We need the following fact.

**Proposition 3.2.** *Let* $G \curvearrowright X$ *be a properly discontinuous, quasi-convex cocompact, and isometric action on a* $\mathrm{CAT}(-1)$*-space* $X$*. If the Hausdorff dimension and topological dimension of the limit set* $\Lambda(G) \subseteq \partial_\infty X$ *are both equal to* $n \in \mathbb{N}$*, then* $\Lambda(G)$ *has a weak tangent* $S$ *bi-Lipschitz equivalent to* $\mathbb{R}^n$*.*

*Proof.* It was pointed out in the proof of [1, Theorem 1.2] that the space $\Lambda(G)$ as equipped with a visual metric is Ahlfors $n$-regular, and that the induced action $G \curvearrowright \Lambda(G)$ satisfies the hypotheses of [1, Theorem 1.1]. As was seen in the proof of [1, Theorem 1.1], the space $\Lambda(G)$ then has a weak tangent bi-Lipschitz equivalent to $\mathbb{R}^n$. $\square$

Our final ingredient is Kirchheim's Metric Differentiation Theorem [5, Theorem 2] that in our terminology can be formulated as follows.

**Proposition 3.3.** *Let* $Z$ *be a metric space bi-Lipschitz equivalent to* $\mathbb{R}^n$*. Then* $Z$ *has a weak tangent isometric to* $\mathbb{R}^n$ *equipped with a metric induced by a norm.*

## 4. PROOF OF THEOREM 1.1

By Proposition 3.2 the space $\Lambda(G)$ has a weak tangent $S$ bi-Lipschitz equivalent to $\mathbb{R}$. By Kirchheim's Metric Differentiation Theorem it follows that some weak tangent $S'$ of $S$ is isometric to $\mathbb{R}$; therefore $\Lambda(G)$ itself has a weak tangent isometric to $\mathbb{R}$. By Proposition 3.1 there exist points $\zeta \in \Lambda(G)$, $q \in X$ and an isometry $f\colon \mathbb{R} \to \Lambda(G) \setminus \{\zeta\}$. Here $\Lambda(G) \setminus \{\zeta\}$ is equipped with the restriction of the parabolic metric $d_{\zeta,q}$. In particular, $f$ is Möbius. Consider the hyperbolic plane in the upper half-space model. Then $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, and $\mathbb{R}$ equipped with the Euclidean metric can be considered as the boundary of $\mathbb{H}^2$ equipped with a parabolic metric. If we define $f(\infty) := \zeta$, then $f$ extends to a a homeomorphism from $\partial_\infty \mathbb{H}^2$ onto $\Lambda(G)$. We denote this extension by $\tilde{f}$.

The cross-ratio is independent of the choice of the visual or parabolic metric as we have seen in Lemma 2.3. Therefore, if $\partial_\infty \mathbb{H}^2$ and $\Lambda(G)$ are equipped with visual metrics, then $\tilde{f}$ will preserve the cross-ratios of all 4-tuples of points in $\mathbb{R} = \partial_\infty \mathbb{H}^2 \setminus \{\infty\}$. A limiting argument then implies that $\tilde{f}$ preserves the cross-ratios of *all* 4-tuples of distinct points. Hence $\tilde{f}$ is Möbius.

An application of Proposition 2.4 shows that $\tilde{f}$ is a boundary map of an isometry of $\mathbb{H}^2$ onto a totally geodesic subspace $Y \subseteq X$. In particular, $\partial_\infty Y = \Lambda(G)$, and so $Y$ is the convex hull of $\Lambda(G)$. The result follows. $\square$

## 5. THE CASE WHEN $n \geq 2$

In this section we outline a proof of Theorem 1.1 without the dimensional restriction (note that this generalizes [2, 0.3 Théorème, ($\mathbb{H}^n$ case)]). We begin the proof as in the $n = 1$ case: Again by Propositions 3.1–3.3, there exists a point $\zeta \in \Lambda(G)$ such that $\Lambda(G) \setminus \{\zeta\}$ equipped with a parabolic metric is isometric to a normed vector space $(\mathbb{R}^n, \|\cdot\|)$.

The proof given for $n = 1$ shows that if $R$ is a subset of $(\mathbb{R}^n, \|\cdot\|)$ isometric to $\mathbb{R}$, then $R \cup \{\zeta\}$ forms the boundary of a subset of $X$ isometric to $\mathbb{H}^2$. It follows that the union of all geodesics in $X$ with one endpoint equal to $\zeta$ and the other endpoint in $\Lambda(G) \setminus \{\zeta\}$ is equal to a convex subset $Y \subseteq X$. From the definition of the parabolic visual metric, it is not difficult to verify that the distance function on $Y$ is isometric to the distance function of a Finsler manifold which is an exponential warped product of $(\mathbb{R}^n, \|\cdot\|)$ with $\mathbb{R}$. If $y \in Y$, then blowing up the pointed space $(Y, y)$, we get a Gromov-Hausdorff limit which is CAT(0) and isometric to a normed space. It it well-known (and follows immediately from the flat strip theorem) that a normed space can be CAT(0) only if its norm is Euclidean, i.e., comes from an inner product. Thus $Y$ is isometric to a *Riemannian* exponential warped product of $\mathbb{R}^n$ (with the usual Riemannian metric) and $\mathbb{R}$, and is therefore isometric to $\mathbb{H}^{n+1}$. ☐

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