Math 115 Calculus Final
15 December 1997

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: Please read the instructions on each individual problem carefully, and indicate answers as directed. Use units wherever appropriate.

This exam consists of 12 questions on 16 pages (including this cover sheet and a blank final page), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Problem	Points	Score
1	9	
2	4	
3	8	
4	6	
5	4	
6	4	
7	10	
8	6	
9	8	
10	11	
11	15	
12	15	
Total	100	

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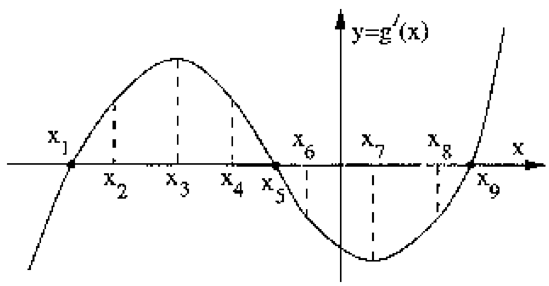
NO PARTIAL CREDIT SECTION. (Problems 1–6.) No explanation necessary; no need to show work.

1. Suppose that the graph below is a graph of $g'(x)$. In other words, this graph is **not** a graph of g; it is a graph of the **derivative** of g.



(a) (3 points) List all of the labelled values of x at which g has a local maximum.

$\boxed{x_5}$

since $g'(x) > 0$ if $x < x_5$, $g'(x) < 0$ if $x > x_5$.

(b) (3 points) List all of the labelled values of x at which g has an inflection point.

$\boxed{x_3, \ x_7}$

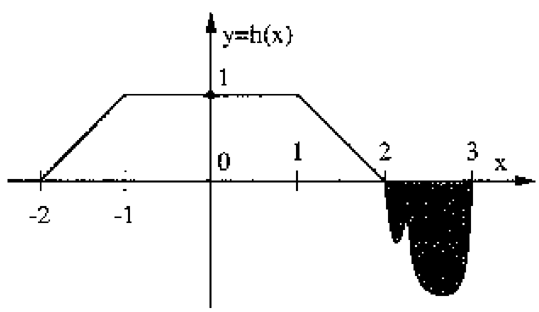
need $g''(x) = 0 \Rightarrow$ tangent to graph horizontal

(c) (3 points) List all of the labelled values of x at which the graph of g is concave up.

$\boxed{x_1, x_2, x_8, x_9}$

need $g''(x) > 0 \Rightarrow$ slope of tangent to graph positive.

2. (4 points) Let $h(x)$ be the function graphed below, and suppose that the shaded region has area 1. Find the average value of $h(x)$ over the interval $-2 \le x \le 3$.



The average value is: $\boxed{2/5}$

$$\text{Average} = \frac{1}{5}\int_{-2}^{3} h(x)\,dx = \frac{1}{5}\int_{-2}^{2} h(x)\,dx + \frac{1}{5}\int_{2}^{3} h(x)\,dx = \frac{1}{5}(3) + \frac{1}{5}(-1) = \frac{2}{5}$$

3. Let f and g be functions described by the following table of data.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-3	1	3	-5.03	-2.16
-1	3	0	-1.97	-1.25
1	0	-1	-1.51	1.06
3	-1	1	-1.33	2.02

(a) (4 points) Let $h(x) = f(g(x))$. Compute $h'(1)$, rounding your final answer to 2 decimal places, if necessary.

$$h'(x) = f'(g(x))\, g'(x) \Rightarrow$$

$$h'(1) = f'(g(1))\, g'(1) = f'(-1)\, g'(1)$$

$$= -1.97 * 1.06 = -2.0882$$

Answer: $\boxed{-2.09}$

(b) (4 points) Let $m(x) = g(x)\cos(2x)$. Compute $m'(-3)$, rounding your final answer to 2 decimal places, if necessary.

$$m'(x) = g'(x) \cos 2x + g(x) \frac{d}{dx} \cos 2x$$

$$= g'(x) \cos 2x + g(x)[-2 \sin 2x]$$

$$m'(-3) = g'(-3) \cos(-6) + g(-3)[-2 \sin(-6)]$$

$$= (-2.16)(.9602) + 3(-2)(.2794)$$

$$= -3.7504$$

Answer: $\boxed{-3.75}$

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4. (6 points) Let f, g, and h be functions described by the table below. Suppose that one of the functions is linear, one of the functions is exponential, and the remaining function is neither linear nor exponential. Identify the linear function and the exponential function.

x	$f(x)$	$g(x)$	$h(x)$
1	2.0713	0.6202	1.7141
2	8.2852	1.0000	3.7854
3	18.6417	1.6124	5.8567
4	33.1408	2.6000	7.9280
5	51.7825	4.1924	9.9993

$\boxed{h}$ is the linear function.

$3.7854 - 1.7141 = 2.0713$

$5.8567 - 3.7854 = 2.0713$

$\boxed{g}$ is the exponential function.

$1/.6202 = 1.6124$

$1.6124/1 = 1.6124$

5. (4 points) At the beginning of an experiment, 100 grams of the exponentially decaying radioactive substance Dilithium-182 are placed in a container, and 36 hours later, 25 grams of Dilithium-182 remain in the container. How many hours after the beginning of the experiment were there 50 grams of Dilithium-182 in the container? (Round your answer to the nearest hour, if necessary.)

$P(t) =$ amount after t hours

$P(t) = 100\left(\frac{1}{4}\right)^{t/36}$

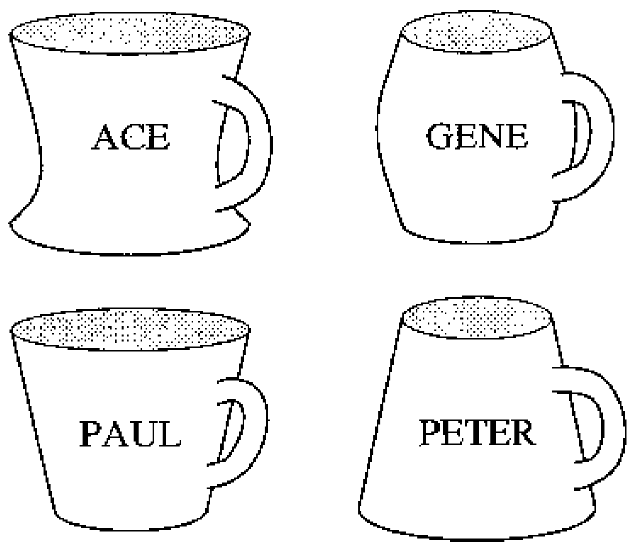
since $P(0) = 100$, $P(36) = \frac{100}{4} = 25$.

Solve $50 = 100\left(\frac{1}{4}\right)^{t/36} \Rightarrow$

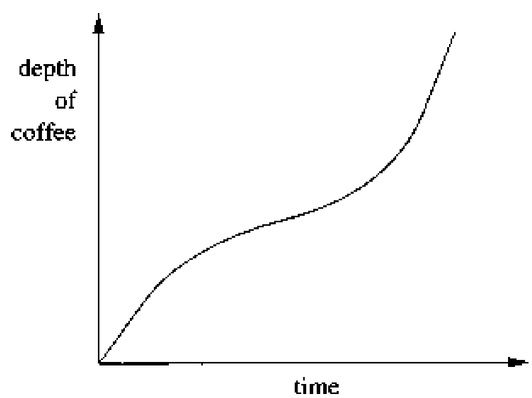
$\frac{1}{2} = \left(\frac{1}{4}\right)^{t/36} \Rightarrow t/36 = \frac{1}{2} \Rightarrow t = 18$

Answer: 18 hours

6. (4 points) Ace, Gene, Paul, and Peter own the coffee mugs pictured below. (The handles are solid and hold no coffee.)



One of them fills his coffee cup at a constant rate (i.e., constant volume per unit time), and the graph of the depth of the coffee against time looks like:



Who filled his coffee cup: Ace, Gene, Paul, or Peter?

GENE	**filled his coffee cup.**

because graph is initially concave implying the cross section increases from the bottom. Later the graph is convex implying the cross section then decreases.

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SHORT ANSWER SECTION. (Problems 7–11.) Limited partial credit may be possible, and a few sentences of explanation may be required.

7. For both parts of this question, you do not need to explain your answer, but show all of your work. Consider the curve given by

$$x^2y - x^2 = 12e^{y-4}.$$

(a) (7 points) Find the equation of the tangent line to the curve at $(2, 4)$.

$$\frac{d}{dx}\left(x^2y - x^2\right) = \frac{d}{dx}\left(12e^{y-4}\right) \Rightarrow$$

$$x^2\frac{dy}{dx} + 2xy - 2x = 12e^{y-4}\frac{dy}{dx}.$$

Put $x = 2,\ y = 4 \Rightarrow$

$$4\frac{dy}{dx} + 16 - 4 = 12\frac{dy}{dx} \Rightarrow$$

$\frac{dy}{dx} = \frac{3}{2}$, Eqn. tangent is $y - 4 = \frac{3}{2}(x-2)$

$\Rightarrow y = \frac{3}{2}x + 1$

Answer: $y = \frac{3}{2}x + 1$

(b) (3 points) Using the tangent line to the curve at $(2, 4)$, approximate the value of y at $x = 1.9$.

$$y = \frac{3}{2}(1.9) + 1 = 3.85$$

Answer: 3.85

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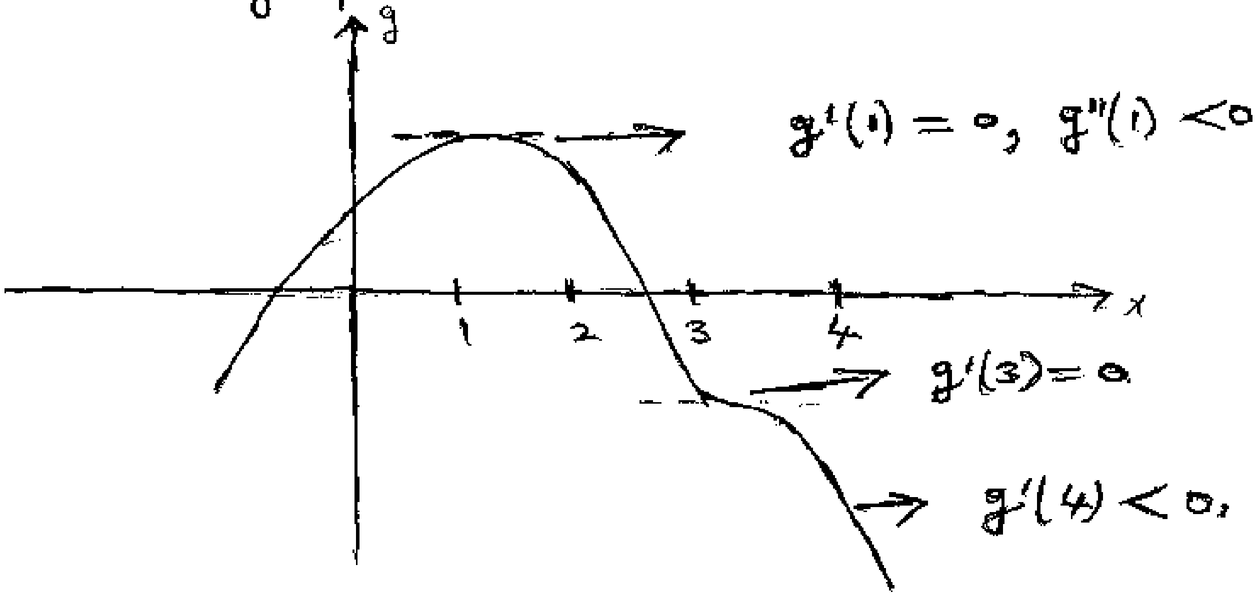
8. (6 points) Suppose that g is differentiable everywhere, and furthermore, that:

- g has precisely two critical points, at $x = 1$ and $x = 3$.
- $g'(4) = -1$.
- $g''(1) = -2$.

At $x = 1$, does g have a **global** minimum, a **global** maximum, or neither? **EXPLAIN** your answer briefly in complete sentences. A graph may be useful.

graph looks like



g has a global maximum at $x = 1$, because: (a) $g'(1) = 0$, $g''(1) < 0 \Rightarrow$ $x = 1$ is a local maximum.

(b) $g'(x) > 0$ if $x < 1$ since $g''(1) < 0$ $\Rightarrow g'(x) > 0$ for $x < 1$ and x close to 1 but g' can only change sign at $x = 1, 3$.

(c) $g'(4) < 0 \Rightarrow g'(x) < 0$ if $x > 3$.

(d) $g'(x) < 0$ if $1 < x < 3$ since $g''(1) < 0$ and g' only changes sign at $x = 1, 3$.

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9. (8 points) For $f(x) = x^4 - .0002x^2$, and $-3 \le x \le 1$:

(a) Find the value(s) of x for which $f(x)$ has a **global** maxmium, and **EXPLAIN** briefly how you know $f(x)$ reaches its global maxmium at this point/these points.

(b) Find the value(s) of x for which $f(x)$ has a **global** minimum, and **EXPLAIN** briefly how you know $f(x)$ reaches its global minimum at this point/these points.

You can do this by completing the square. Thus

$$f(x) = [x^2 - .0001]^2 - (.0001)^2 .$$

Hence global maximum when $|x^2 - .0001|$ is largest ie $x = -3$ in the interval $[-3, 1]$.

Hence answer to (a) is $x = -3$ is global maximum.

(b) global minimum is at $x^2 - .0001 = 0$

$$\Rightarrow \quad x = \pm\sqrt{.0001} = \pm .01$$

since $[x^2 - .0001]^2$ is always nonnegative.

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10. This question has parts (a), (b), and (c), on two pages.

Pepa is parachuting from an airplane. Let t be the number of seconds after she first jumps, and let $v(t)$ be her **downwards** velocity (in meters per second) at time t. 3 seconds after she jumps (i.e., at $t = 3$ seconds), she opens her parachute, and for $3.7 \le t \le 4.1$, she records the following data.

t (in seconds)	3.7	3.8	3.9	4.0	4.1
$v(t)$ (in meters per second)	11.22	10.67	10.02	9.29	8.56

ASSUME that for $3.7 \le t \le 4.1$, her velocity is always decreasing.

(a) (4 points) Estimate Pepa's acceleration at $t = 4.0$ seconds, as accurately as possible from the given data. You do not need to explain your answer, but show all your work, and state your final answer in the form of a complete sentence, using the correct units.

acceleration at $t = 4 = v'(4)$,

$$v'(4) \simeq \frac{v(4+h) - v(4)}{h}$$

$$h = .1 \Rightarrow v'(4) \simeq \frac{v(4.1) - v(4)}{.1} = \frac{8.56 - 9.29}{.1}$$

$= -7.3$. Her deceleration is 7.3 meters per seconds^2 at $t = 4$ secs.

(b) (4 points) As accurately as possible given only the information above, give upper and lower estimates of the total distance Pepa falls between $t = 3.7$ seconds and $t = 4.1$ seconds. You do not need to explain your answer, but show all your work, and state your final answer in the form of a complete sentence, using the correct units.

Since her velocity always decreases

upper bound is $.1[11.22 + 10.67 + 10.02 + 9.29]$

$= .1 * 41.2 = 4.12$ m.

lower bound is

$.1[10.67 + 10.02 + 9.29 + 8.56]$

$= .1 * 38.54 = 3.854$ m.

(c) (3 points) Briefly **EXPLAIN** in complete sentences why the actual total distance Pepa falls between $t = 3.7$ seconds and $t = 4.1$ seconds is between your two answers in (b).

Her speed in the period between 3.7 & 3.8 seconds is less than 11.22 and 10.67. It is 11.22 at the beginning of the 10th of a second and decreases to 10.67 at the end of the 10th of a second because velocity is always decreasing.
Hence the distance gone in this period of 1/10 of a second is less than $.1 * 11.22 = 1.122$ m and larger than $.1 * 10.67 = 1.067$ m. Similarly with the other periods.

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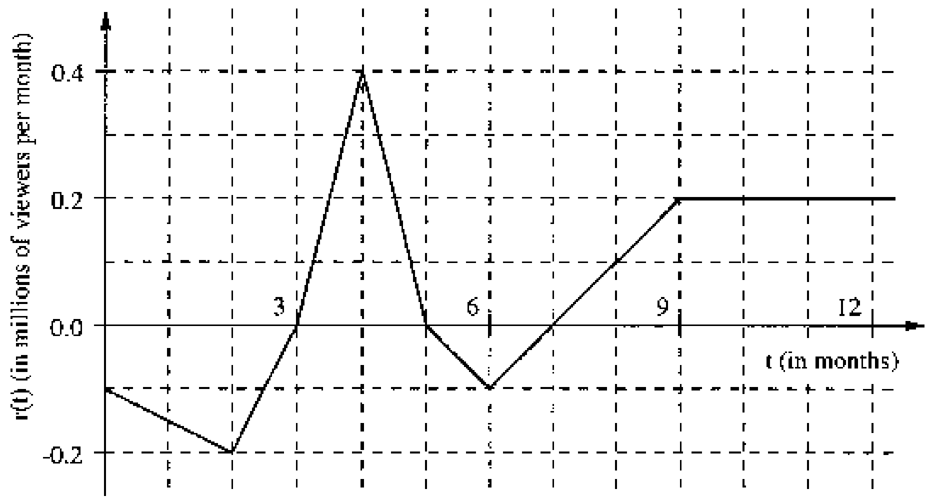
11. This question has parts (a), (b), and (c), on three pages.

The everything-but-music-videos television channel MTTV has been keeping track of how its regular viewership is **changing**. At the beginning of 1995 ($t = 0$ months), 5.6 million people were watching MTTV regularly. Let $r(t)$ be the **rate of change** of MTTV's regular viewership, in millions of viewers per month, t months after the beginning of 1995, and suppose the graph of $r(t)$ is as shown below.



(a) (4 points) At what time(s) in 1995 was MTTV's viewership **decreasing** at the fastest rate? **JUSTIFY** your answer using no more than **TWO SENTENCES.**

MTTV's viewership is decreasing at the fastest rate at $t = 2$ ie at the beginning of March. The reason is $t = 2$ is the lowest point on the rate graph.

(b) (7 points) At what time(s) in 1995 were the **fewest** people watching MTTV regularly, and how many people were watching MTTV regularly then? Briefly **JUSTIFY** your answer.

The candidates for the times the fewest people are watching are when the rate graph crosses the axes from negative to positive i.e. $t = 3$ or $t = 7$.

number of people watching at $t = 3$ is in millions,

$5.6 - 0.4 = 5.2$.

number of people watching at $t = 5$ is

$5.2 + 0.4 = 5.6$

number of people watching at $t = 7$ is

$5.6 - .1 = 5.5$.

Hence minimum number watching is at $t = 3$ and the number is 5.2 million.

(c) (4 points) At what time(s) in 1995 were there 6 million people watching MTTV regularly? You do not need to explain your answer, but show all your work, and state your final answer in the form of a complete sentence, using the correct units.

at $t = 9$ there are

$5.5 + .2 = 5.7$

at $t = 10$ there are

$5.7 + .2 = 5.9$.

$t = 10\frac{1}{2}$ there are

$5.9 + .1 = 6$.

Hence 6 million people are watching when $t \geq 10\frac{1}{2}$ months.

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ESSAY QUESTION

EXPLAIN your answers using complete sentences. Use graphs, tables, and/or formulas in your explanations, if possible. If you use a graph in your answer, it must be drawn neatly and carefully, and labelled properly.

12. This question has parts (a), (b), and (c), on two pages.

Farmer Fiona wants to hire workers to pick 900 bushels of apples. Each worker can pick 5 bushels per hour and is paid \$2.00 per bushel. Fiona must also pay a supervisor \$20.00 per hour while the picking is in progress, and she also has overhead costs of \$16.00 per worker hired.

(a) (2 points) If Fiona hires w workers, how many hours will it take them to pick all 900 bushels of apples? No explanation necessary, but show all your work.

number of hours $= \frac{900}{5w} = \frac{180}{w}$.

(b) (5 points) Find a formula for Fiona's total costs C as a function of the number of workers w she hires. No explanation necessary, but show all your work.

$$C(w) = 2 * 900 + 20\left(\frac{180}{w}\right) + 16w$$

since $2 * 900$ = cost of picking at \$2 per bushel.

$20\left(\frac{180}{w}\right)$ = cost of supervisor

$16w$ = overhead costs for workers.

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(c) (8 points) How many workers should Fiona hire in order to minimize her total costs?
Make sure that you **EXPLAIN** in complete sentences how you know that the number of workers in your final answer **minimizes** Fiona's costs.

$$C'(w) = -\frac{20 \ast 180}{w^2} + 16$$

$$C'(w) = 0 \Rightarrow w^2 = \frac{20 \ast 180}{16}$$

$$\Rightarrow w = \frac{6 \ast 10}{4} = 15$$

$$0 < w < 15 \Rightarrow C'(w) < 0$$

$$w > 15 \Rightarrow C'(w) > 0.$$

Hence $w = 15$ is a global minimum in region $0 < w < \infty$.

Fiona should hire 15 workers to minimise her costs.