Math 115 Calculus Exam I
February 4, 1998

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: This test consists of 10 questions on 10 pages (including this cover sheet and a blank final page), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show all your work! On questions 5–10 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	5	
2	6	
3	8	
4	12	
5	7	
6	10	
7	10	
8	10	
9	14	
10	18	
Total	100	

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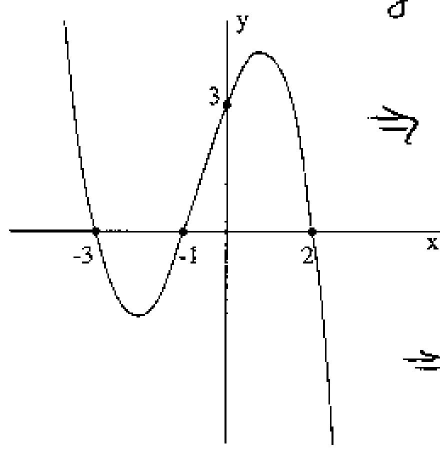
NO PARTIAL CREDIT SECTION. (Problems 1–4.) No explanation necessary; no need to show work.

1. (5 points) Find a possible equation for the function $f(x)$ described by the graph below. (Note that the graph is not necessarily drawn to scale.)



y has zeros at $x = -3$, $x = -1$, $x = 2$

$\Rightarrow y = c\,(x+3)(x+1)(x-2)$ for some constant c.

$x = 0 \Rightarrow y = 3.$

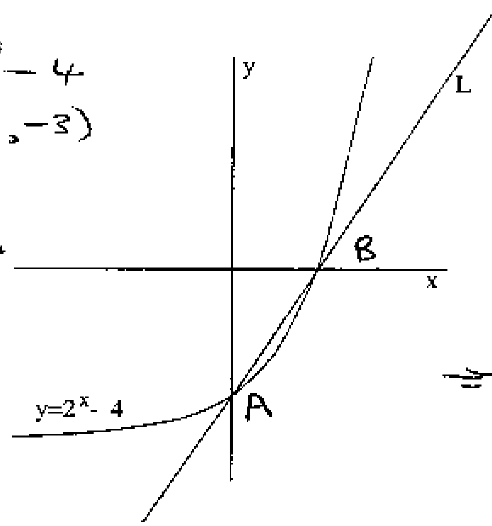
$\Rightarrow 3 = c\,3\,(1)(-2) \Rightarrow c = -\frac{1}{2}$

Answer: $y = -\frac{1}{2}(x+3)(x+1)(x-2)$

2. (6 points) Suppose that the curve below is the graph of $y = 2^x - 4$. Find the exact equation of the line L shown below. (Note that the graph is not necessarily drawn to scale.)

A is point $x = 0$, $y = 2^0 - 4$ ie. $A = (0, -3)$

B is point $y = 0$, $0 = 2^x - 4$ ie. $B = (2, 0)$



L passes through A & B $\Rightarrow$ Eqn. L is

$$\frac{y+3}{x-0} = \frac{0+3}{2-0}$$

$$\Rightarrow y + 3 = \frac{3}{2}x$$

Answer: $y = \frac{3}{2}x - 3$

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3. (5 points) **Find the equations** of the horizontal asymptote(s) and vertical asymptote(s) to the graph of the function $y = \frac{(2x-1)(x+3)}{100x^2-1}$.

Vertical: Need $y \Rightarrow \infty$ as x approaches a finite value $\Rightarrow 100x^2-1=0 \Rightarrow$
$x = \pm \frac{1}{10}$.

Horizontal: Need y approach a finite value as $x \Rightarrow \infty$. When x is large
$y \simeq \frac{(2x)(x)}{100x^2} = \frac{1}{50} = .02$

Vertical (of the form $x = a$): $x = .1$ and $x = -.1$

Horizontal (of the form $y = b$): $y = .02$

4. (12 points) The table below gives values for the functions $f(x)$ and $g(x)$.

x	f(x)	g(x)
1	5	8
2	2	7
3	4	12
4	1	3
5	3	9

$f(4) = 1 \Rightarrow g(f(4)) = g(1) = 8$

Put $x = f^{-1}(3) \Rightarrow f(x) = 3$

$\Rightarrow x = 5.$

$h(3) = f(5) = 3$

(a) Find $g(f(4))$.

(b) Find $f^{-1}(3)$

(c) If $h(x) = f(x+2)$, find $h(3)$.

Answers: $g(f(4)) =$ 8

$f^{-1}(3) =$ 5

$h(3) =$ 3

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SHORT ANSWER SECTION. (Problems 5–9.) Limited partial credit may be possible, and a few sentences of explanation may be required. In addition, you may be asked to state your final answer in the form of a complete sentence. Remember to show your work!

5. (7 points) The following table of data gives values of a function $f(x)$.

x	0.4	0.8	1.2	1.6	2.0
f(x)	2.56	5.12	10.24	30.72	92.16

Is $f(x)$ an exponential function? Explain your answer in two or three complete sentences.

Suppose $f(x) = Ae^{bx}$. Then

$f(.4) = Ae^{.4b}$, $f(.8) = Ae^{.8b}$

$f(1.2) = Ae^{1.2b}$, $f(1.6) = Ae^{1.6b}$

$f(2.0) = Ae^{2b}$. Hence

$$\frac{f(.4)}{f(.8)} = e^{(.4-.8)b} = e^{-.4b}$$

$$\frac{f(.8)}{f(1.2)} = e^{(.8-1.2)b} = e^{-.4b}$$

$$\frac{f(1.2)}{f(1.6)} = e^{(1.2-1.6)b} = e^{-.4b}.$$

Now $\frac{f(.4)}{f(.8)} = \frac{2.56}{5.12} = .5$

$\frac{f(.8)}{f(1.2)} = \frac{5.12}{10.24} = .5$; $\frac{f(1.2)}{f(1.6)} = \frac{10.24}{30.72} = .333$

f is not exponential since these 3 ratios are not all equal.

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6. (10 points) Suppose that a group of freshman is making a large batch of chili for the Hunger Coalition. The chili needs to be ready in 3 hours. The amount of chili they can prepare is proportional to the square root of the number of students helping. If 8 students are helping they can produce 15 quarts of chili. What is the smallest number of students needed to make 22 quarts of chili? Give your answer in the form of a complete sentence.

Let s = number of students helping

q = amount of chili made.

Then $q = k\sqrt{s}$

$$15 = k\sqrt{8} \Rightarrow k = \frac{15}{\sqrt{8}}$$

To get number of students needed for 22 quarts use

$$22 = k\sqrt{s} \Rightarrow$$

$$s = \frac{22^2}{k^2} = 22^2\left(\frac{\sqrt{8}}{15}\right)^2$$

$$= \frac{22^2 \cdot 8}{15^2} = \frac{3872}{225}$$

$$= 17.21$$

17 students are needed to make the 22 quarts of chili in 3 hours.

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7. (10 points) In both parts of this question you should give your answer in one or two complete sentences.

(a) (5 points) Suppose that $f(x)$ is the number of thousands of copies of the new Allyam Brothers record sold in the first x days since its release. Explain the meaning of $f^{-1}(16) = 5$ in practical terms.

$f^{-1}(16) = 5 \Rightarrow f(5) = 16$

Hence in 5 days 16000 records are sold.

(b) (5 points) Patti Smith is running for city council in Ann Arbor. Her campaign manager, Ted Nugent, asserts that the number of votes she will receive is a function of the amount she spends on campaign ads. Let $f(x)$ be the number of votes she will receive if she spends x dollars on campaign ads. Explain the meaning of $f(3000) = 2000$ in practical terms.

If she spends \$3000 she will receive 2000 votes.

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8. (10 points) Suppose that the population of zebra mussels in the Great Lakes is growing exponentially and that there were 3 times as many zebra mussels in the Great Lakes on July 1, 1997 as there were on January 1, 1991. How many months does it take the population of zebra mussels in the Great Lakes to double? State your answer in a complete sentence (Round the doubling time to the nearest tenth of a month.)

There are $6*12+6 = 78$ months from Jan 1, 1991 to July 1, 1997. Since population triples in that time, one has $P = P_0 3^{t/78}$, where t is number of months since Jan 1, 1991 and P_0 is population on Jan 1, 1991.

For the population to double have

$$2P_0 = P_0 3^{t/78} \Rightarrow 2 = 3^{t/78} \Rightarrow$$

$$\ln 2 = \ln 3^{t/78} = \frac{t}{78}\ln 3 \Rightarrow$$

$$t = \frac{78 \ln 2}{\ln 3} = 49.21$$

It takes 49.2 months for the population to double.

9. (14 points) At Pescadero Beach in California there is a footpath joining the beach to a small island with beautiful tidal pools. At low tide the water level is 2 feet below the footpath, while at high tide the water level is 10 feet above the footpath. The first low tide of the year occurs at noon on January 1, 1998 and the next high tide is at 7pm on January 1, 1998.

(a) (8 points) Assuming that the height of the water is modelled by a trigonometric function, find a formula for the height of the water above the footpath. Let h be the height, in feet, and let t be the number of hours since noon on January 1, 1998.

The period $= 2 * 7 = 14$ hours

amplitude $= \frac{1}{2}(10+2) = 6$.

$t = 0$ corresponds to low tide $\Rightarrow$

$$h = -6 \cos \frac{2\pi t}{14} + k$$

$t = 0 \Rightarrow h = -2 \Rightarrow k = 4$

Hence $h(t) = -6 \cos \frac{\pi t}{7} + 4$

Answer: $\boxed{h(t) = -6 \cos \frac{\pi t}{7} + 4}$

(b) (6 points) How many times is there a high tide between noon on January 1, 1998 and noon of February 1, 1998? (Recall that there are 31 days in January.) Give your answer in a complete sentence.

High tide every 14 hours beginning at 7 p.m. on Jan. 1st.

Number of hours from 7 p.m. on Jan. 1st to noon on Feb 1st

$= 24 * 30 + 17 = 737$ hours

$$\frac{737}{14} = 52.64$$

Hence there are 53 high tides.

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ESSAY QUESTION

10. (18 points) Marin County has recently decided to secede from the union. Newly elected President Boxer is concerned because she has recently observed that the population is growing exponentially, while the number of people who can be fed by local farmers is only growing linearly. She has collected the following data.

Year	Population	Number of people who can be fed
1987	100,000	105,000
1989	108,000	115,650
1991	116,640	126,300
1993	125,971	136,950
1995	136,049	147,600
1997	146,933	158,250

As her recently appointed Secretary of Agriculture she has asked you to prepare a report which predicts how long Marin County will continue to be able to feed all its citizens.

EXPLAIN your prediction in a brief essay using complete sentences. Use graphs (labelled carefully and neatly), tables, and/or formulas wherever possible. You may continue on the next page, which has been left blank for this purpose.

$\frac{108,000}{100,000} = 1.08 \qquad \frac{116,640}{108,000} = 1.08$ etc

Population $P = 100,000\,(1.08)^{t/2}$, where $t =$ number of years since 1987.

$115,650 - 105,000 = 10650$
$126300 - 115,650 = 10650$ etc.

$Q = 105,000 + (10,650)\frac{t}{2}$,
$Q =$ population which can be fed, t is number of years since 1987.

$Q(0) = 105,000 > 100,000 = P(0)$
Since P grows exponentially and Q only linearly P will become larger than Q eventually

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We can continue the table on the previous page

Year	Population	People can be fed
1998	152697	163575
1999	158687	168900
2000	164913	174225
2001	171382	179550
2002	178106	184875
2003	185093	190200
2004	192354	195525
2005	199900	200850
2006	207742	206175

To get population increase multiply by $(1.08)^{1/2}$. To get increase in number of people that can be fed add $\frac{1}{2}(10,650) = 5325$

The number of people that can be fed is larger than the population until the year 2005. After that it is smaller.