Introduction to Interest Rate Modeling

Presentation by: Holly Chung

University of Michigan

March 5, 2009
Zero Coupon Bonds

- A zero coupon bond with maturity date $T$ is a contract which guarantees the holder $1$ to be paid on the date $T$.

\[ P(t, T) = \text{Value at time } t \text{ of } 1 \text{ received at time } T \]
Zero Coupon Bonds

- Interest rates are stochastic $\Rightarrow P(t, T)$ unknown until time $t$
- $P(t, T)$ is a function of two variables: initiation time $t$ and maturity time $T$
- Dependence on $T$ reflects term structure of interest rates (interest rate vs. time to $T$)
- Convention: Present time $t = 0$

$\Rightarrow$ Initial observable is $P(0, T)$ for all $T$
Yield-to-Maturity

The yield-to-maturity (or yield) is defined implicitly by:

\[ P(t, T) = e^{-R(t, T)(T-t)} \]

\[ \Rightarrow R(t, T) = -\frac{\log P(t, T)}{T - t} \]

It is the unique constant short-term interest rate implied by the market bond price \( P(t, T) \).
Instantaneous Forward Rate

The *instantaneous forward rate* $f(t, T)$ is defined by

$$P(t, T) = e^{- \int_t^T f(t, \tau) d\tau}$$

It is the deterministic time-varying interest rate describing all loans starting at time $t$ with various maturities.

$$f(t, T) = - \frac{\partial \log P(t, T)}{\partial T}$$
Short Rate

The *instantaneous short-term interest rate* $r(t)$ is given by

$$r(t) = f(t, t)$$

i.e. the rate earned on the shortest-term loans starting at time $t$. 
Goal of Interest Rate Modeling

To price and hedge *interest rate derivatives*

*What’s an interest rate derivative?*
A financial instrument of which the underlying asset is the right to pay or receive an amount of money at a given interest rate. It has the largest derivatives market in the world.

Examples:
- Interest rate swap (fixed for floating)
- Interest rate cap and floor
Modeling Interest Rates

Three viewpoints:

- Simple Short Rate Models (e.g. Vasicek Model)
- Richer Short Rate Models (e.g. Hull-White Model)
- Forward Rate Models (e.g. HJM Model)
Example: Vasicek Model

- The (risk-neutral) process for $r$ is:

$$dr(t) = [\theta - ar(t)] dt + \sigma dW(t)$$

where $\theta$, $a > 0$, and $\sigma$ are constants.

- Advantage: Explicit formulas for $P(t, T)$ and $R(t, T)$.

- Disadvantage: Too few parameters $\Rightarrow$ no hope of calibrating to the entire yield curve $P(0, T)$.

- As a result, Vasicek and similar short rate models are rarely used in practice.
Example: Hull-White Model

- The (risk-neutral) process for $r$ is:

\[ dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t) \]

where $a$ and $\sigma$ are still constant but $\theta$ is a function of time $t$.

- Advantages:
  1. Under some conditions, the Hull-White model correctly reproduces the entire yield curve $P(0, T)$ at time 0.
  2. It still leads to explicit formulas for $P(t, T)$ and $R(t, T)$.

- Disadvantage: gives little freedom in modeling evolution of the yield curve.
Example: Heath, Jarrow, and Morton Model

- The process for instantaneous forward rate $f$ is:
  
  $$ df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t) $$

- Advantage: gives the user complete freedom in choosing the volatility term structure

- Disadvantage:
  1. No guidance on how to choose $\sigma(t, T)$
  2. Difficult to implement this model numerically.
Conclusion

In interest rate modeling, there is a tradeoff between simplicity and accuracy.