Hirota’s Bilinear Method
(The Direct Method)

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MATH 651 Final Presentation
Outline

• The Direct Method
• Motivating Example – Korteweg-de Vries (KdV) Equation
• Mechanics of the Direct Method
• Soliton solutions to the Nonlinear Schrödinger Equation (NLS)
• Physical Applications – the Benjamin-Ono Equation
• Soliton solutions to the Benjamin-Ono Equation
The Direct Method, The Hirota Method

- Developed in parallel to the Inverse Scattering Transform
  - C. 1970
  - Direct Method (Japan), Hirota/Bilinear Method (everywhere else)
- Seeks to directly find the form of solitons solutions admitted by a given nonlinear PDE
  - Develops a framework applicable to many nonlinear PDEs
Inverse Scattering vs. Direct Method

\[ u_t + 6uu_x + u_{xxx} = 0 \] (KdV)

Overall goal

Direct Scattering Transform

Inverse Scattering Transform

Explicit Time Evolution

"Bilinear Ansatz"

Bilinear Equations

Soliton Solutions
Example of the Direct Method: Soliton Solution to KdV

• Hirota’s original discovery of the exact solution by bilinearization

\[ u_t + 6uu_x + u_{xxx} = u_t + 3(u^2)_x + u_{xxx} = 0 \]

\[ u = 2(\log(v))_{xx} = 2 \left( \frac{v_x}{v} \right)_x \]

\[ 2 \left( \frac{v_x}{v} \right)_{xt} + 12 \left( \frac{v_x}{v} \right)^2_x + 2 \left( \frac{v_x}{v} \right)_{xxx} = 0 \]

\[ v_{xt} - v_x v_t + v v_{xxxx} - 4v_{xxx} v_x = C = 0 \]

\[ v = \Sigma_n \epsilon^n v_n; v_0 = 1 \]

\( \epsilon^1: (v_1)_{xt} + (v_1)_{xxxx} = 0 \Rightarrow v_1 = \exp(\eta); \eta = P_1 x + \Omega t + \eta_0, P_1^3 + \Omega = 0 \]

\( \epsilon^2: \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} \right)(2v_2 + v_1 \ast v_1) = 0 \]

\[ \Rightarrow v = 1 + \epsilon v_1 \]
Soliton Solution to KdV (cont.)

• This method yields the single soliton solution quite simply, despite the relatively simple choice of the perturbation series

• Interactions between solitons can be determined systematically by taking linear superpositions in the first order of the perturbation
  • This equation is linear, so everything is still satisfied
  • This will require higher-order matching, but everything is still done systematically, based on how many solitons we want to have interacting

\begin{align*}
v &= 1 + \epsilon v_1 = 1 + \epsilon \exp\{\eta\} \\
u &= 2 \log(1 + \epsilon \exp\{\eta\}) \\
u &= \frac{2P_1^2 \exp\{\eta\}}{(1 + \exp\{\eta\})^2} \\
u &= \frac{p_1^2}{2} \text{sech}^2 \frac{\eta}{2}
\end{align*}
Single Soliton Solution to KdV
Single Soliton Solution to KdV (cont.)

- Frequency content of the single soliton
2-Soliton Solution to KdV

• Following the intuition of the single soliton solution, take the same ansatz for KdV

\[ v_{xt}v - v_xv_t + vv_{xxxx} - 4v_{xxx}v_x = C = 0 \]
\[ v = \Sigma_n \epsilon^n v_n; v_0 = 1 \]
\[ \epsilon^1: (v_1)_{xt} + (v_1)_{xxxx} = 0 \Rightarrow v_1 = \exp(\eta_1) + \exp(\eta_2); \eta_i = P_i x + \Omega_i t + \eta_0i, P_i^3 + \Omega = 0 \]
\[ v = 1 + \epsilon(\exp(\eta_1) + \exp(\eta_2)) + \epsilon^2 a_{12} \exp(\eta_1 + \eta_2); a_{12} = \frac{(P_1 - P_2)^2}{(P_1 + P_2)^2} \]
\[ u = \frac{v_{xx}v - v_x^2}{v^2} \]
2-Soliton Solution to KdV (cont.)
2-Soliton Solution to KdV (cont.)
N-Soliton Solution to KdV

• Given the 2-soliton solution, we can guess what will happen for \( N \geq 2 \)
• Essentially, we will have terms like individual solitons, and interaction terms for when solitons are located in the same place in spacetime

\[
f_1 = \Sigma_n \exp(\eta_n) \\
f = \Sigma_{\mu=0,1} \exp(\Sigma_j \mu_j \eta_j + \Sigma_{k<j} \mu_j \mu_k A_{jk}) \\
\exp(A_{jk}) = \frac{(p_j - p_k)^2}{(p_j + p_k)^2}
\]

• This is Hirota’s original result in 1971
Mechanics of the Direct Method

• Solving KdV:
  • Assume a form which yields a bilinear equation
  • Solve the new equation using a proper perturbation series
  • First order evolution equation is linear, yielding soliton solutions
  • Higher order terms can be used to balance interactions between solitons
  • Eventually, high-order terms can be set to 0, giving a convergent series

• Hirota’s Formulation
  • Assume a bilinear form of the solution
  • Introduce a new operator, defining a class of Hirota equations
  • Several Hirota equations have been solved directly
    • All other Hirota equations for which we have an answer obtained by transform
Hirota Derivatives

- Define the Hirota Derivative by MacLaurin Series:

\[ L\{\varphi, \partial_t, \partial_x \} \varphi = s(x, t) = 0 \]

\[ \varphi = f(x) \circ g(x) = f(x + y) \circ g(x - y) \bigg|_{y=0} \]

\[ f(x + y) \circ g(x - y) = \Sigma_n \left( \frac{1}{n!} \right) D_x^n (f \circ g) y^n \]

- Evaluate the first few terms by hand:

\[ \varphi = f(x + y) \circ g(x - y) \bigg|_{y=0} = \Sigma_n \left( \frac{1}{n!} \right) \frac{\partial^n}{\partial y^n} (f \circ g) y^n \]

\[ f \circ g \bigg|_{y=0} = f(x) g(x) + \left( \frac{\partial f}{\partial y} g - f \frac{\partial g}{\partial y} \right) y + \frac{1}{2} \left( \frac{\partial^2 f}{\partial y^2} g - 2 \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial y^2} \right) y^2 + \ldots \]

\[ f \circ g \bigg|_{y=0} = f(x) g(x) + \left( \frac{\partial f}{\partial x} g - f \frac{\partial g}{\partial x} \right) y + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} g - 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} \right) y^2 + \ldots \]

\[ D_x^1 (f \circ g) = D_x (f \circ g) = \frac{\partial f}{\partial x} g - f \frac{\partial g}{\partial x}, D_x^2 = \left( \frac{\partial^2 f}{\partial x^2} g - 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} \right) \]

\[ D_x^n (f \circ g) \equiv \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x) \circ g(x') \]
Hirota Equations

• Define a Hirota Equation in the following way:

\[ P(D_q^i)(f \circ f) = 0 \]

• Key Properties
  • Hirota Equations are explicitly bilinear (e.g. quadratic in function)
  • Hirota Equations which do not depend on a constant \( D_q^0 \) admit constants as solutions
  • Hirota Equations which do not depend on a constant \( D_q^0 \) explicitly admit pseudo-soliton (or better) solutions

• Several famous nonlinear equations can be represented as a Hirota equations
  • KdV
  • Nonlinear Schrödinger Equations (both Focusing and Defocusing)
  • Toda Equations
  • Benjamin-Ono
Pseudo-Soliton Solutions

• Assume a Hirota Equation with no constant derivative

\[ P(\Sigma_{m>0,n}D_{x_n}^m)(f \circ f) = 0 \]
\[ f = 1 + \Sigma_{n \geq 1} \epsilon^n f_n \]
\[ \epsilon^0: P(1 \circ 1) = 0 \]
\[ \epsilon^1: P(\Sigma_{m>0,n} \partial_{x_n}^m)f_1(1) = 0; \partial_{x_n}^m := \frac{\partial^m}{\partial x_n^m} \]

• The first order of the equation is satisfied by a soliton solution

\[ f_1 = \Sigma_j \exp\{\Sigma_i k_i^j x_i\} \]
\[ \Sigma_i k_i^j = 0 \]

• Represents a sum over j solitons, traveling in a sum over i dimensions

• For a series that eventually converges, these solitons are usually conserved
KdV by Hirota’s Method

• Making the substitution before, KdV yields a Hirota equation

\[ u_t + 6uu_x + u_{xxx} = 0 \]
\[ u = 2(\log(v))_{xx} = \left(\frac{v_x}{v}\right)_x \]
\[ v_{xt}v - v_xv_t + vv_{xxxx} - 4v_{xxx}v_x = C = 0 \]
\[ (D_x D_t + D_x^4)(v \circ v) = 0 \]

• Given the structure of the Hirota equation, this equation clearly admits soliton solutions
Nonlinear PDE’s Expressed as Hirota Equations

• Defocusing and Focusing NLS
\[ i\Psi_t + \Psi_{xx} + 2c|\Psi|^2\Psi = 0; c = \pm 1 \]
\[ (iD_t + D_x^2)(G \circ F) = \lambda (G \circ F) \] (1)
\[ D_x^2(F \circ F) - 2c|G|^2 = \lambda F^2 \]

• Toda Lattice
\[ \frac{\partial^2}{\partial t^2} \log(1 + V_n(t)) = V_{n+1}(t) - 2V_n(t) + V_{n-1}(t) \]
\[ V_n(t) = \frac{\partial^2}{\partial t^2} \log(f_n(t)) = \frac{f_{n+1}(t)f_{n-1}(t)}{f_n(t)^2} - 1 \]
\[ \left(D_t^2 - 4 \sinh^2(\frac{D_n}{2})\right)(f_n \circ f_n) = 0 \]

• Benjamin-Ono
  • Note: H is a Hilbert transform
\[ u_t + 4uu_x + Hu_{xx} = 0 \]
\[ (iD_t - D_x^2)(f' \circ f) = 0 \]
Nonlinear Schrödinger Equation: Another Application

• NLS can be bilinearized using the following transformation

\[ i\Psi_t + \Psi_{xx} + 2c|\Psi|^2\Psi = 0; c = \pm 1 \]

\[ \Psi = \frac{G}{F} \]

\[ (iD_t + D_x^2)(G \circ F) = \lambda (G \circ F) \quad (1) \]

\[ D_x^2(F \circ F) - 2c|G|^2 = \lambda F^2 \]

• This assumption yields what is called an envelope soliton

• The structure of the soliton depends on the choice of parameter \( \lambda \)
  • Bright soliton: \( \lambda = 0 \) (FNLS)
  • Dark soliton: \( \lambda = 1 \) (DNLS)
Soliton Solutions to FNLSE: “Bright Solitons”

- Choosing Focusing form on NLS yields Bright Solitons

\[ \psi = \frac{A_1 \text{sech}(\xi_1) \exp(i\zeta_1)(\cos(\phi_1 + i\sin(\phi_1) \tanh(\xi_2)) + A_2 \text{sech}(\xi_2) \exp(i\zeta_2)(\cos(\phi_2 + i\sin(\phi_2) \tanh(\xi_1)))}{\cosh(a) + \sinh(a)(\tanh(\xi_1) \tanh(\xi_2) - \text{sech}(\xi_1) \text{sech}(\xi_2) \cos(\xi_1 - \zeta_2)} \]

\[ a = \log \left( \frac{|P_1 - P_2|}{|P_1 + P_2^*|} \right) \]

\[ \phi_1 = \arg \left( \frac{P_1 - P_2}{P_1 + P_2^*} \right), \phi_2 = \arg \left( \frac{P_2 - P_1}{P_2 + P_1^*} \right) \]

\[ A_i = \frac{1}{\sqrt{2}} |P_i + P_i^*|, \xi_i = \text{Re}(P_ix - \Omega_i t + C_i), \zeta_i = \text{Im}(P_ix - \Omega_i t + C_i) \]

\[ \Omega_i = -\frac{i}{2} P_i^2 \]
Soliton Solution to FNLSE
Soliton Solution to FNLSE
2-Soliton Solution to FNLSE (Collision)
2-Soliton Solutions to FNLSE
Advantages of solving FNLSE using Direct Method

- Soliton solutions can be found over the course of a day
  - Uses a very common ansatz in Direct Method to bilinearize FNLSE
  - Bilinearized equations can be solved perturbatively or by making a soliton ansatz

- Parameters in the Soliton solution are clear
  - Amplitude(s)
  - Speed(s)
  - Location of Soliton center(s)
Limitations of Direct Method in FNLSE

• Does not solve a general problem for a given nonlinear PDE
  • Soliton solutions say nothing about the initial conditions
  • Soliton solutions do not necessarily form a complete basis for evolving an initial condition

• Using language learned in class based on Inverse Scattering Transform
  • These solutions come from the reflectionless initial condition
  • Riemann-Hilbert Problem is solved by placing simple complex poles based on parameters in solution

• Solitary wave solution is not guaranteed to say anything about higher-order poles in the complex plane or an initial condition that is not reflectionless
The Benjamin-Ono Equation

• Another integrable equation:
  • H is the Hilbert Transform
  \[ u_t + 4uu_x + Hu_{xx} = 0 \]

• Describes the motion of internal waves in the deep ocean
  • Provides environmental mismatch for sonars
  • Serves to toss around submersible ships
  • Provides some pitch and toss, shaping the deep ocean ecosystem

• Describes the propagation of Rossby waves in a rotating fluid
Bilinearization of Benjamin-Ono Equations

• Assume the following form, for $f$ is a linear equation in space

\[ u_t + 4uu_x + Hu_{xx} = u_t + 2(u^2)_x + Hu_{xx} \]

\[ u = \frac{i}{2} \frac{\partial}{\partial x} \log \left( \frac{f^*}{f} \right) \quad H := P \left( \int_R \frac{u(\tau)}{t-\tau} \right) \]

\[ Hu = \frac{i}{2} H \frac{\partial}{\partial x} \log \left( \frac{f^*}{f} \right) = \frac{i}{2} H \left( \frac{f'^*}{f} - \frac{f'}{f} \right) \]

\[ Hu = \frac{i}{2} H \left( \frac{1}{f^*} - \frac{1}{f} \right) = \frac{i}{2} \left( \frac{1}{f^*} + \frac{1}{f} \right) = \frac{1}{2} \frac{\partial}{\partial x} \log(f^*f) \]

\[ \frac{i}{2} \log \left( \frac{f^*}{f} \right)_{xt} + 2 \left( \frac{i}{2} \log \left( \frac{f^*}{f} \right) \right)_x - \log(f^*f)_{xxx} = 0 \]

\[ i(f_t^*f - f^*f_t) - (f_{xx}^*f - 2f_x^*f_x + f_{xx}f) = 0 \]

\[ (iD_t - D_x^2)(f^* \circ f) = 0 \]
Soliton Solution of the Benjamin-Ono Equation

• 1\textsuperscript{st} Class of soliton solutions: Normalizable solitons

\[(iD_t - D_x^2)(f^* \circ f) = 0\]
\[f = \sum_n \epsilon^n f_n\]
\[\epsilon^1: (iD_t - D_x^2)(f_1^* \circ f_0 + f_0 \circ f_1) = 0\]
\[i \left( \frac{\partial f_1^*}{\partial t} - \frac{\partial f_1}{\partial x} \right) - \left( \frac{\partial^2 f_1^*}{\partial x^2} + \frac{\partial^2 f_1}{\partial x^2} \right) = 0\]
\[f_1 = i(x - at - x_0) + \frac{1}{a} = i\theta(x, t) + \frac{1}{a}\]
\[\epsilon^2: (iD_t - D_x^2)(f_2^* \circ f_0 + f_0 \circ f_2 + f_1^* \circ f_1) = 0\]
\[(iD_t - D_x^2)(f_1^* \circ f_1) = 0 \Rightarrow f_2 = 0\]
\[f = \epsilon f_1 + f_0\]
\[f_0 = 0\]
\[u = \frac{i}{2} \left[ \log \left( \frac{f^*}{f} \right) \right]_x \Rightarrow u = \frac{a}{1 + a^2 \theta(x, t)^2}\]
Behavior of the Soliton Solution of Benjamin-Ono
Frequency Content of the Soliton
2-Soliton Solution of the Benjamin-Ono Equation

- 2-Soliton solution can be found by making the same assumption as in the 1-soliton case, the structure is the same as finding the 2-soliton solution to KdV

\[
\begin{align*}
  f_2 &= c_3 \theta_1 \theta_2 + c_2 \theta_2 + c_1 \theta_1 + c_0 \\
  f_2 &= -\theta_1 \theta_2 + i \left( \frac{\theta_1}{a_2} + \frac{\theta_2}{a_1} \right) + \frac{1}{a_1 a_2} \left( \frac{a_1 + a_2}{a_1 - a_2} \right)^2 \\
  u &= \frac{i}{2} \left( \log \left( \frac{f_2^*}{f_2} \right) \right)_x \\
  u &= \frac{\left( a_1^2 a_2 \theta_1 + a_2^2 a_1 \theta_2 \right) \left( \theta_2 + \theta_1 \right) + \left( \frac{a_1 + a_2}{a_1 - a_2} \right)^2 - a_1 a_2 \theta_1 \theta_2 \left( a_1 + a_2 \right)}{\left( \frac{a_1 + a_2}{a_1 - a_2} - a_1 a_2 \theta_1 \theta_2 \right)^2 + \left( a_1 \theta_1 + a_2 \theta_2 \right)^2}
\end{align*}
\]
2 Soliton Solution
N-Soliton Solutions of the Benjamin-Ono Equation

• As a curiosity, the 2-Soliton solution can be expressed as the following determinant:

\[ f_2 = \det \begin{pmatrix} i\theta_1 + \frac{1}{a_1} & 2 \\ 2 & \frac{a_1 - a_2}{a_1 - a_2} \\ \frac{a_1 - a_2}{a_1 - a_2} & i\theta_2 + \frac{1}{a_2} \end{pmatrix} \]

• Similarly, the N-Soliton solution (presented here without proof) can be written:

\[ f_N = \det M_N \]

\[ \{M_N\}_{jk} = \begin{cases} i\theta_j + \frac{1}{a_j}, & j = k \\ \frac{2}{a_j - a_k}, & j \neq k \end{cases} \]
Review of Topics Covered

• One-, Two-, and N-Soliton solutions of several integrable systems
  • Shallow Water Waves (KdV)
  • Wave Bullet/Focusing Media (Focusing Nonlinear Schrödinger Equation)
  • Deep Ocean Internal Wave Travel (Benjamin-Ono Equation)

• Bilinearization of wide class of nonlinear PDE’s
  • Three possible bilinear ansatzes to make (most popular choices)
  • Ways to derive soliton expressions
  • General form of multi-soliton solutions, once a single soliton has been found

• General pros/cons of the Direct Method
Conclusion

• The Direct Method (also known as the Hirota Method) is a viable way of finding solitary wave solutions to a wide class of nonlinear, integrable partial differential equations
  • Allows direct exploration of the exact form of solitary waves
  • Assumptions made in solving differential equations tend to involve less rigorous assumptions than inverse scattering

• The Direct Method does not guarantee a solution to the general problem
  • Solutions are obtained making an assumption
  • Solitary wave solutions usually do not form a complete basis in which to evolve an initial condition
References


