Fourier-like method for solution of linearized Korteweg-de Vries Equation

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Math 651

April 17, 2018
Consider the KdV equation

$$q_t + q_{xxx} - 6qq_x = 0$$  \hspace{1cm} (1)

Let $q(x, t)$ be a solution and consider a perturbation $q + u$ for $u$ small:

$$(q + u)_t + (q + u)_{xxx} - 6(q + u)(q + u)_x \approx u_t + u_{xxx} - 6(qu)_x$$  \hspace{1cm} (2)

How to find exact solution to the initial value problem

$$u_t + u_{xxx} - 6(qu)_x = 0$$

$$u(x, 0) = u_0(x)$$  \hspace{1cm} (3)

Analysis of solution concerns stablity of $q$ as a solution to KdV.

Sachs [4] found a Fourier-like transform involving Jost solutions valid for wide class of initial conditions!
The Cauchy integral formula

\[ f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \]  

(4)

for complex valued functions. What about operator valued functions?

\[ f(T) = \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{\lambda - T} d\lambda \]  

(5)

where \( T \) is an operator on a complex Banach space \( X \). For the moment, let’s suppose \( X \) is finite dimensional.

**Definition**

The spectrum \( \sigma(T) \) of the operator \( T \in L(X) \) is the set of complex numbers \( \lambda \) for which the operator \( \lambda I - T \) is not invertible. On \( \sigma(T)^c \), denote \( R(\lambda; T) = (\lambda I - T)^{-1} \) the resolvent. The index \( \nu(\lambda) \) is the smallest non-negative integer for which \( (\lambda I - T)^\nu x = 0 \) for all \( x \in \ker(\lambda I - T)^{\nu+1} \).
Theorem

If $P$ and $Q$ are polynomials, $P, Q : L(X) \to L(X)$, then

$P(T) = Q(T) \iff (P - Q)(\lambda) : \mathbb{C} \to \mathbb{C}$ has a zero of order at least $\nu(\lambda)$ at each point $\lambda \in \sigma(T)$.

Proof.

The minimum polynomial of $T$ is $m_T(\lambda) = (\lambda - \lambda_1)^{\nu(\lambda_1)} \cdots (\lambda - \lambda_n)^{\nu(\lambda_n)}$ for $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$.

Next an important concept:

Definition

Denote $\mathcal{F}(T)$ the family of functions which are analytic on some open covering of $\sigma(T)$. The covering may be dependent on the function.
**Theorem**

In finite dimensions, \( \{ f(T) \mid f \in \mathcal{F}(T) \} \) is the finite dimensional vector space of polynomials in \( T \)

**Proof.**

Suppose for each \( \lambda_i \in \sigma(T) \), \( f \) has a Taylor series

\[
f(\lambda) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda_i)}{n!} (\lambda - \lambda_i)^n.
\]

Then there exists a polynomial \( P(\lambda) \) of degree \( \leq \sum_{\lambda_i \in \sigma(T)} (\nu(\lambda_i) - 1) \) such that \( P^m(\lambda_i) = f^m(\lambda_i) \) for \( m = 0, 1, \ldots, \nu(\lambda_i) - 1 \). In this case, the Weierstrass factorization theorem implies

\[
\frac{f(\lambda) - P(\lambda)}{m_T(\lambda)} \in \mathcal{F}(T) \Rightarrow f(T) = P(T)
\]

wherever \( f \) exists even though \( P(\lambda) \neq f(\lambda) \) for \( \lambda \not\in \sigma(T) \)

Images of analytic functions are defined only up to so many terms in their Taylor series.
Example: Let $T = I$ and $f$ be analytic on a neighborhood of one and define $p(\lambda) = f(1) = 1$. Then $f(I) - p(I) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\lambda_i)}{n!}(I - I)^n = 0$.

Define $e_{\lambda_0}(\lambda)$ being one on a neighborhood of $\lambda_0$ and zero in a neighborhood of $\sigma(T) \setminus \{\lambda_0\}$.

Projections: $E(\lambda_0) \equiv e_{\lambda_0}(T)$
- $E(\lambda_0) \neq 0 \iff \lambda_0 \in \sigma(T)$,
- $E(\lambda_0)E(\lambda_1) = 0$ if $\lambda_0 \neq \lambda_1$ and $E(\lambda_0)^2 = E(\lambda_0)$, and
- $\sum_{\lambda_i \in \sigma(T)} E(\lambda_i) = 1$.

Jordan normal form decomposition:
- $X = E(\lambda_1)X \oplus \cdots \oplus E(\lambda_n)X$
- $E(\lambda_i)X = \{x \in X|(T - \lambda_iI)^{\nu(\lambda_i)}x = 0\}$
- $f(T) = \sum_{\lambda_i \in \sigma(T)} \sum_{n=0}^{\nu(\lambda_i)-1} \frac{f^{(n)}(\lambda_i)}{n!}(T - \lambda_iI)^n E(\lambda_i)$
Back to Cauchy integral formula.

Suppose \( f \in \mathcal{F}(T) \) is defined on an open set \( U \supset \sigma(T) \) and for \( \lambda \not\in \sigma(T) \), define \( r(\xi) = (\lambda - \xi)^{-1} \) on \( U \).

\[
\Rightarrow r(T) = \sum_{\lambda_i \in \sigma(T)} \sum_{n=0}^{\nu(\lambda_i) - 1} \frac{(T - \lambda_i I)^n}{n!(\lambda - \lambda_i)^{n+1}} E(\lambda_i)
\]

so that

\[
\frac{1}{2\pi i} \int_{\partial U} f(\lambda) r(T) d\lambda = \frac{1}{2\pi i} \int_{\partial U} f(\lambda) \sum_{\lambda_i \in \sigma(T)} \sum_{n=0}^{\nu(\lambda_i) - 1} \frac{(T - \lambda_i I)^n}{n!(\lambda - \lambda_i)^{n+1}} E(\lambda_i) d\lambda
\]

\[
= \frac{1}{2\pi i} \sum_{\lambda_i \in \sigma(T)} \sum_{n=0}^{\nu(\lambda_i) - 1} \frac{(T - \lambda_i I)^n}{n!} \int_{\partial U} \frac{f(\lambda)}{(\lambda - \lambda_i)^{n+1}} E(\lambda_i) d\lambda
\]

\[
= \sum_{\lambda_i \in \sigma(T)} \sum_{n=0}^{\nu(\lambda_i) - 1} \frac{(T - \lambda_i I)^n}{n!} f^{(n)}(\lambda_i)
\]

\[
= f(T)
\]
Infinite dimensions, $T$ a bounded operator

**Definition**

The resolvent set $\rho(T)$ is the set of complex numbers $\lambda$ for which the resolvent $R(\lambda; T) = (\lambda I - T)^{-1}$ exists as a bounded operator with domain $X$. The spectrum $\sigma(T) = \rho(T)^c$.

**Lemma**

$\rho(T)$ is open, $R(\lambda; T)$ is analytic on $\rho(T)$ and $\sup \sigma(T) \leq \|T\|$.

**Proof.**

Let $\lambda \in \rho(T)$ and $\mu < \|R(\lambda; T)\|^{-1}$. Then $\|\mu R\| < 1$ and

$$
\sum_{k=0}^{\infty} (-\mu)^k R(\lambda; T)^{-(k+1)} \equiv S(\mu) = ((\mu + \lambda)I - T)^{-1}
$$

showing $\lambda + \mu \in \rho(T)$ and $R(\lambda + \mu; T) = S(\mu)$ is analytic at $\mu = 0$. 

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Let $U$ be a neighborhood of $\sigma(T)$ and suppose $f(\lambda)$ is analytic on $U$, i.e., $f \in \mathcal{F}(T)$. It will now be a matter of definition that

$$f(T) = \frac{1}{2\pi i} \int_{\partial U} f(\lambda)R(\lambda; T)d\lambda$$

(8)

**Definition**

A point $\lambda_0 \in \sigma(T)$ is isolated if there exists a neighborhood $U$ of $\lambda_0$ such that $U \cap \sigma(T) = \{\lambda_0\}$. An isolated point $\lambda_0$ is said to be a pole of $T$ of order $\nu(\lambda_0)$ if it is a pole of order $\nu(\lambda_0)$ of $R(\lambda; T)$. The set $\sigma_p(T)$ of $\lambda \in \sigma(T)$ such that $\lambda I - T$ is not injective is called the point spectrum. The set $\sigma_c(T)$ of $\lambda \in \sigma(T)$ such that $\lambda I - T$ is injective, $(\lambda I - T)X$ is dense in $X$ but $(\lambda I - T)^{-1}$ is not bounded is called the continuous spectrum.

Notice $\lambda I - T$ is not injective implies there is an $x \in X$ such that $Tx = \lambda x$. Thus $\lambda \in \sigma_p(T) \iff \ker(\lambda I - T)$ is nontrivial.
Theorem (Spectral Mapping Theorem)

If \( f \in \mathcal{F}(T) \), then \( f(\sigma(T)) = \sigma(f(T)) \).

Notice the above theorem says that if \( f(T) = 0 \), then \( f(\sigma(T)) = 0 \) but it does not imply that \( f(\lambda) \equiv 0 \) on every neighborhood of \( \sigma(T) \).

Theorem (Minimal Equation)

Let \( f, g \in \mathcal{F} \). Then \( f(T) = g(T) \) if and only if \( f(\lambda) = g(\lambda) \) on a neighborhood of \( \sigma(T) \) or, if \( \sigma \equiv \{\lambda_1, \ldots, \lambda_k\} \) is a collection poles of order \( \nu(\lambda_i) \), a neighborhood of \( \sigma(T) \setminus \sigma \). In this case, \( f - g \) must have a zero of order at least \( \nu(\lambda_i) \).

Indicator functions play same role:

- \( e_{\lambda_0}(T) = \frac{1}{2\pi i} \int_{C_{\lambda_0}} (\lambda I - T)^{-1} d\lambda = 0 \Leftrightarrow \lambda \neq \lambda_0 \)
- \( e_{\lambda_0}(\lambda)e_{\lambda_1}(\lambda) \equiv 0 \Rightarrow e_{\lambda_0}(T)e_T(\lambda) = 0 \) by above theorem, and
- \( e_{\lambda_0}(\lambda)^2 = e_{\lambda_0}(\lambda) \Rightarrow e_{\lambda_0}(\lambda)(e_{\lambda_0}(\lambda) - 1) \equiv 0 \Rightarrow e_{\lambda_0}(T)(e_{\lambda_0}(T) - 1) = 0 \Rightarrow e_{\lambda_0}(T)^2 = e_{\lambda_0}(T) \)
Proof of Minimal Equation.

(Sketch) WLOG, set $g(T) = 0$.

$(\Rightarrow)$ Let $f(z) = 0$ on an open set containing $\sigma(T) \setminus \sigma$;

$$f(T) = \frac{1}{2\pi i} \sum_{i=1}^{k} \int_{C_i} f(\lambda)R(\lambda; T)d\lambda$$  \hspace{1cm} (9)

Apply Cauchy’s theorem: $f$ has a zero at $\lambda_i$ of at least order $\nu(\lambda_i) \Rightarrow f(T) = 0$

$(\Leftarrow)$ Suppose $f(\lambda)$ is analytic on a neighborhood $U \supset \sigma(T)$ s.t. $f(T) = 0$. Use Compactness of $\sigma(T)$ (boundedness of $T$) to show $f(\lambda) \equiv 0$ on neighborhoods of accumulation points of $\sigma(T)$ since $\sigma(f(T)) = \sigma(0) = 0 = f(\sigma(T))$. If $\lambda_1$ isolated and $f \equiv 0$ on every nbhd of $\lambda_1$, done. Suppose $f(\lambda) \not\equiv 0$ around $\lambda_1$. Spectral mapping shows $\exists n > 0$ s.t. $\lambda_1$ is a zero of order $n$ of $f$. Therefore, $(\lambda - \xi)^n/f(\xi)$ is analytic $\Rightarrow (\lambda_1 I - T)^k e_{\lambda_1}(T) = \frac{(\lambda - T)^k}{f(T)} f(T) = 0$ for all $k \leq n$ implying $\lambda_1$ is a pole of order at most $n$ of $T$. 

$\square$
Unbounded, closed operators

**Definition**

Let the domain of \( T \mathcal{D}(T) \subset X \) a Banach space. An operator is said to be closed if its graph \((x, Tx)\) is closed, i.e., if \( x_n \to x \) and \( Tx_n \to y \), then \( x \in \mathcal{D}(T) \) and \( Tx = y \).

Now \( \mathcal{F}(T) \) will denote the set of functions analytic on a neighborhood of \( \sigma(T) \) and at complex infinity.

How to get \( f(T) \) using the Cauchy integral for unbounded \( \sigma(T) \)?

Map to a bounded set: let \( \alpha \in \rho(T) \) and define \( A = (T - \alpha I)^{-1} \) and \( \mu = \Phi(\lambda) = (\lambda - \alpha)^{-1} \).

**Lemma**

*For* \( \alpha \in \rho(T) \), \( \Phi(\sigma(T) \cup \infty) = \sigma(A) \) *and each* \( \phi \in \mathcal{F}(A) \) *is in one-to-one correspondence with an* \( f \in \mathcal{F}(T) \) *via* \( \phi(\mu) = f(\Phi^{-1}(\mu)) \). *It follows by boundedness of A that* \( \sigma(T) \) *is closed.*
**Definition**

For \( f \in \mathcal{F}(T) \), define \( f(T) = \phi(A) \) where \( \phi(\mu) = f(\Phi^{-1}(\mu)) \)

The meaning for the above definition is that since \( \alpha \in \rho(T) \) and \( \rho(T) \) is open, \( \Phi(\sigma(T) \cup \infty) = \sigma(A) \) is bounded. Hence \( \phi(\mu) \) can be defined as a contour integral enclosing \( \sigma(T) \), from which \( f \) can be recovered.

**Theorem**

If \( f \in \mathcal{F}(T) \), then \( f(T) \) is independent of \( \alpha \in \rho(T) \). Let \( V \supset \sigma(A) \) be open with boundary \( \Gamma \) being the finite union of Jordan arcs such that \( f \) is analytic on \( V \cup \Gamma \). Let \( \Gamma \) have positive orientation w.r.t. the possibly unbounded set \( V \). Then

\[
 f(T) = f(\infty)I + \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)R(\lambda; T) d\lambda
\]  

(10)

**Proof.**

Notice the integral is independent of \( \alpha \). In fact, since \( f(\lambda)R(\lambda; T) \) is analytic on \( \rho \), we may assume \( \alpha \not\in V \cup \Gamma \). Integral then follows from change of variables.
The KdV equation has a Lax pair as $q$ is a solution of (1) is equivalent to

$$\frac{dL}{dt} = [B, L] \quad (11)$$

where

$$L = -\partial_x^2 + q, \quad B = -4\partial_x^3 + 3q\partial_x + 3\partial_x q \quad (12)$$

In light of (11) the spectrum of $L$ is time-invariant. This in turn motivates the auxiliary problem at time $t=0$:

$$L\psi = -\partial_x^2 \psi + q(x, t)\psi = \lambda \psi \quad (13)$$
If \( \| (1 + |x|)q(x) \|_1 < \infty \), \( \sigma_p(L) = \{ \lambda \in \mathbb{R} : L\psi = \lambda \psi \text{ and } \lambda < 0 \} \) is finite

Setting \( \lambda = k^2 \), if \( \text{Im} \, k = 0 \):

- Jost solutions satisfy \( (L - k^2 I)f_\pm = 0 \) and form a basis of solutions with boundary data \( f_\pm(x, k) \sim e^{\pm ikx} \) as \( x \to \pm \infty \)
- \( \| f_\pm \|_{L^2(\mathbb{R})} = \| (L - \lambda I)^{-1}0 \|_{L^2(\mathbb{R})} = \infty \)
- \( f_\pm \) correspond to \( \sigma_c(L) \)

If \( \text{Im} \, k > 0 \):

- \( L\psi = k^2 \psi \Rightarrow \psi(x) \sim e^{-\sqrt{-\lambda} |x|} \) as \( |x| \to \infty \) a possibility
- Jost solution are a single \( L^2 \) eigenfunction iff their Wronskian=0
- \( T(k) \) has simple poles in this case
- If \( \psi \in L^2 \) so assume \( \| \psi \|_2 = 1 \). Then there are unique \( c_k \) s.t.
  \[ \lim_{x \to \infty} \psi(x)e^{ikx} = c_k \]
\( f_+(x, k) \) a solution for \( \Rightarrow \Im k = 0 \Rightarrow f_+^* \) (complex conjugate) a solution

Wronskian \([f_+, f_+^*] = 2ik\) implies \( f_+ \) and \( f_+^* \) form a basis.

There exists \( a(k) \) and \( b(k) \) s.t. for \( f_-(x, k) \sim e^{-ikx} \) as \( x \to -\infty \),

\[
f_- = af_+^* + bf_+
\]  

(14)

By Cramer’s rule,

\[
a(\lambda) = \frac{[f_+, f_-]}{[f_+, f_+^*]} = \frac{[f_+, f_-]}{2ik}
\]

(15)

\[
b(\lambda) = \frac{[f_-, f_+^*]}{[f_+, f_+^*]} = \frac{[f_-, f_+^*]}{2ik}.
\]

(16)

We can re-write (14) as

\[
T(k)f_+ = f_+^* + R(k)f_+
\]

(17)

where \( T = 1/a \) and \( R = b/a \) are the transmission and reflection coefficients.
The scattering data for KdV is the collection $R(k)$, $\sigma(L)$ and the $c_k$.

Nice properties:

- $\sigma(L)$ is conserved in time
- $c_k(t) = c_k(0)e^{4k^3t}$
- $R(k, t) = R(k, 0)e^{8ik^3t}$, analytic on $\text{Im } k > 0$
- $T(k)$ conserved in time, meromorphic for $\text{Im } k > 0$, simple poles on $\sigma_p(L)$.
- $f_{\pm}(x, k, t) \sim \exp\{\pm i(kx + 4k^3t)\}$ as $x \to \pm \infty$ analytic on $\text{Im } k > 0$
- $|f_{\pm}/e^{\pm ikx} - 1| \leq \exp\{C_1|k|^{-1}\}\frac{C_2}{|k|}$
- $|\partial_x(f_{\pm}/e^{\pm ikx}) - \int_x^\infty e^{\pm 2ik(y-x)}Q(y)dy| \leq C_3(1 + |k|)^{-1}$
- $T(k) = 1 + \mathcal{O}(k^{-1})$ as $|k| \to \infty$.

The solution at a later time $q(x, t)$ is

$$q(x, t) = -2\partial_x K(x, x)$$  \hspace{1cm} (18)

where $K$ satisfies the Gel’fand-Levitan integral equation

$$K(x, y) + B(x + y) + \int_x^\infty B(y + z)K(y, z)dz = 0$$ \hspace{1cm} (19)

where

$$B(\xi) = \sum_{n=1}^N c_n^2 e^{ik_n \xi} + \frac{1}{2\pi} \int_{\mathbb{R}} b(k)e^{ik\xi} dk$$ \hspace{1cm} (20)
The formal adjoint of the linearized KdV equation is

\[ \nu_t + \nu_{xxx} - 6q\nu_x = 0. \]  
(21)

So if \( \nu \) is a solution to (21), we have

\[ \nu_{xt} + \nu_{xxxx} - 6\partial_x(q\nu_x) = 0 \]  
(22)

or by writing \( u = \nu_x \),

\[ u_t + u_{xxx} - 6(Qu)_x = 0, \]  
(23)

the linearized KdV equation. Suffices to solve (21).

If \( \psi \) a solution to the Schrodinger equation, then for each \( t \),

\[ q(x, t) = \frac{\psi_{xx}(x,t) + k^2\psi(x,t)}{\psi(x,t)} \]  
so that

\[ 0 = q_t + q_{xxx} - 6qq_x \]

\[ = \partial_t \left( \frac{\psi_{xx} + k^2\psi}{\psi} \right) + \partial_{xxx} \left( \frac{\psi_{xx} + k^2\psi}{\psi} \right) - 6 \left( \frac{\psi_{xx} + k^2\psi}{\psi} \right) \partial_x \left( \frac{\psi_{xx} + k^2\psi}{\psi} \right) \]  
(24)
This gives \( R \equiv \psi_t + \psi_{xxx} - 3(q + k^2)\psi_x \) satisfies \( LR = k^2R \)

If \( \psi \sim \exp\{ \pm i(kx + 4k^3t) \} \) as \( x \to \pm \infty \), \( R \to 0 \) as \( x \to \pm \infty \) implying \( R \) is the zero solution.

If \( \psi_1 \) and \( \psi_2 \) are (not necessarily linearly independent) solutions of Schrodinger with correct asymptotics, then direct calculation using \( R = 0 \) shows \( \psi = \psi_1\psi_2 \) is a solution to the adjoint operator

\[ \Rightarrow \partial_x(\psi_1\psi_2) \text{ solves linearized KdV}. \]

\( \psi = \psi_1\psi_2 \) also satisfies

\[ T\psi_x \equiv -\frac{1}{4}\psi_{xxx} + q\psi_x + \frac{1}{2}q_x\psi = k^2\psi_x. \quad (25) \]

\( T \) has spectrum of \( L \) but with eigenfunctions being derivatives of quadratic terms. The (formal) resolvent of \( T \) \( r(k) = (k^2I - T)^{-1} \) could then be integrated around a neighborhood of \( \sigma(T) \) to obtain the identity in terms of squared eigenfunctions.

Sachs uses this informal analogy to motivate the coming integral relation:
Theorem

If $\phi \in L^1(\mathbb{R})$ and continuous and if $\int_{\mathbb{R}} (1 + x^2) |q(x, t)| dx < \infty$, then

$$\phi(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{T^2(k)}{4\pi ik} \partial_x \left( f^2_+(x, k)f^2_-(y, k) - f^2_-(x, k)f^2_-(y, k) \right) \phi(y) dy dk$$

$$+ \text{residue terms} \quad (26)$$

where the residue occur from $k$ integration crossing the poles of $T(k)^2$ for $k \in \sigma_p(L)$.

Proof.

(Sketch) Carry out $y$ integration. $f_\pm \to \exp(\pm ikx)$ and $T \to 1$ as $|k| \to \infty$ so deform path of $k$ integration on to a semicircle in the upper half plane. Correct for residue terms at poles of $T$. Resulting integral reads

$$\frac{1}{\pi} \int_{C_R} \left( \int_{\mathbb{R}} e^{2ik|x-y|} \phi(y) dy \right) dk + \mathcal{O}(R^{-1}) \text{ as } R \to \infty \quad (27)$$
In light of last theorem, the solution to the initial value problem

\[ u_t + u_{xxx} - 6(qu)_x = 0 \]
\[ u(x, 0) = u_0(x) \]  

is

\[ u(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{T^2(k)}{4\pi ik} \partial_x f^2_+ (x, t, k)f^2_- (y, 0, k)
\]
\[ - f^2_-(x, t, k)f^2_-(y, 0, k)u_0(y)dydk + \text{residue terms} \]  

(29)
Bibliography

Percy Deift and Eugene Trubowitz.
Inverse scattering on the line.

Nelson Dunford and Jacob T Schwartz.
*Linear operators part I: general theory*, volume 7.

Clifford S Gardner, John M Greene, Martin D Kruskal, and Robert M Miura.
Method for solving the korteweg-devries equation.

Robert L Sachs.
Completeness of derivatives of squared schrödinger eigenfunctions and explicit solutions of the linearized kdv equation.