Math 651

Topic 7: Long Time Asymptotics for the Defocusing Nonlinear Schrödinger Equation by the Steepest-Descent Method

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The Fourier transform was invented precisely to solve initial-value problems for linear equations. For the Schrödinger equation:

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} = 0, \quad \psi(x, 0) = \psi_0(x), \]

the fundamental commutative diagram for the Fourier transform yields a closed-form integral formula for the solution.
For this formula to be useful, information must be extracted from the integrals. Recall the classical tools for asymptotic analysis of integrals:

- Laplace’s method.
- Method of steepest descent.
- Method of stationary phase.

These methods allow us to study several limits of physical interest:

- Long time (scattering, diffraction theory).
- Rough data (Gibbs phenomenon).
- Short waves (geometrical optics, weak dispersion).
Initial-value problems for some nonlinear wave equations can be treated similarly thanks to the inverse scattering transform. Consider the defocusing nonlinear Schrödinger equation:

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi = 0, \quad \psi(x, 0) = \psi_0(x). \]

Here the commutative diagram for the inverse-scattering transform yields a procedure (less explicit, perhaps) for solving the problem.
Introduction

Generalization to nonlinear problems.

To understand the solution of initial-value problems for nonlinear wave equations by inverse-scattering, we may seek \textit{nonlinear analogues} of the classical methods of asymptotic analysis for integrals.
Long-Time Asymptotics for Dispersive Waves
Defocusing NLS equation.

Consider

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi = 0, \quad \text{(Defocusing Cubic Schrödinger)} \]

subject to the initial condition \( \psi(x, 0) = \psi_0(x) : \)

\[ \psi_0(x) = \begin{cases} B, & |x| \leq L, \\ 0, & |x| > L. \end{cases} \]

We are interested in the asymptotic behavior of \( \psi(x, t) \) as \( t \to +\infty \).
Consider for a moment instead the related linear problem

\[
i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} = 0, \quad \psi(x, 0) = \psi_0(x) \quad \text{(Linear Schrödinger)}
\]

We are interested in the asymptotic behavior of \( \psi(x, t) \) as \( t \to +\infty \). By Fourier theory, the solution of this initial-value problem is

\[
\psi(x, t) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{\psi}_0(\lambda) e^{-2i(\lambda x + \lambda^2 t)} \, d\lambda := \lim_{T \to \infty} \frac{1}{\pi} \int_{-T}^{T} \hat{\psi}_0(\lambda) e^{-2i(\lambda x + \lambda^2 t)} \, d\lambda,
\]

where \( \hat{\psi}_0(\lambda) := \mathcal{F}[\psi_0](\lambda) \)

\[
= \frac{B \sin(2L\lambda)}{\lambda}
\]
Long-Time Asymptotics of Dispersive Waves  
Method of stationary phase.

For long-time asymptotics, set $x = ct$ for fixed $c \in \mathbb{R}$, and consider

$$
\psi(ct, t) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{\psi}_0(\lambda) e^{i t I_c(\lambda)} \, d\lambda, \quad I_c(\lambda) := -2(c\lambda + \lambda^2)
$$
in the limit $t \to +\infty$. To approximate $\psi(ct, t)$ for large $t$, use the method of stationary phase (Stokes and Kelvin). Points $\lambda$ of stationary phase satisfy $I'_c(\lambda) = 0$, so there is only one: $\lambda = \lambda_c := -c/2$. Since $I_c(\lambda_c) = c^2/2$ and $I''_c(\lambda_c) = -4$, the stationary phase formula is:

$$
\psi(ct, t) = \frac{1}{\pi} \sqrt{\frac{2\pi}{t|I''_c(\lambda_c)|}} \hat{\psi}_0(\lambda_c) e^{i t I_c(\lambda_c)} - i\pi/4 + O(t^{-3/2})
$$

$$
= \frac{\hat{\psi}_0(\lambda_c)}{\sqrt{2\pi t}} e^{itc^2/2-i\pi/4} + O(t^{-3/2}), \quad t \to +\infty.
$$

Putting $c = x/t$ into the leading term shows that as $t \to \infty$, $\psi(x, t)$ is, up to a phase, a rescaled version of the Fourier transform $\hat{\psi}_0$ of the initial data $\psi_0$.  
This is the classical phenomenon of far-field diffraction.
The stationary phase formula is easily proved for analytic integrands by the classical *method of steepest descent*. Use Cauchy’s Integral Theorem to deform the path of integration \( \lambda \in \mathbb{R} \) into the complex plane on a suitable contour \( C \):

\[
\psi(ct, t) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{\psi}_0(\lambda) e^{itI_c(\lambda)} \, d\lambda = \frac{1}{\pi} \int_{C} \hat{\psi}_0(\lambda) e^{itI_c(\lambda)} \, d\lambda.
\]

The contour \( C \) is selected to pass over critical points of the phase \( I_c(\lambda) \) in the local direction of steepest descent for \( \text{Re}(iI_c(\lambda)) \) (for \( t > 0 \)):
As $I_c(\lambda)$ is quadratic in our case, the approximation is therefore

$$\psi(ct, t) \approx \frac{1}{\pi} \int_{\lambda_c + \Re e^{-i\pi/4}} \hat{\psi}_0(\lambda_c) e^{i t I_c(\lambda)} d\lambda$$

$$= \frac{1}{\pi} \sqrt{\frac{2\pi}{t |I''_c(\lambda_c)|}} \hat{\psi}_0(\lambda_c) e^{i t I_c(\lambda_c) - i\pi/4}.$$

In particular, the sign chart of $\Re (i I_c(\lambda))$ explains the origin of the strange factor of $e^{-i\pi/4}$ appearing in the stationary phase formula. It’s just the angle of steepest descent passage over the saddle point.
Long-Time Asymptotics for Dispersive Waves

Results for nonlinear problem.

The large $t$ limit for the nonlinear problem leads to a formal expansion:

$$\psi(ct, t) \sim t^{-1/2} \left( \alpha(c) + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \frac{(\log(t))^k}{t^n} \alpha_{nk}(c) \right) e^{itc^2/2-i\nu(c)\ln(t)}, \quad t \to \infty.$$ 

All coefficients $\alpha_{nk}(c)$ determined in terms of $\alpha(c)$ by direct substitution. Goal: rigorously determine $\alpha(c)$ in terms of initial data. Key steps:

- In 1976, Segur and Ablowitz [4] related $|\alpha(c)|$ to the initial data via trace formulae.

- In 1976, Zakharov and Manakov [5] showed that the leading term motivates a WKB analysis of the scattering transform $S$, and thereby determined the phase $\text{arg}(\alpha(c))$.


- In 1994, Deift and Zhou [1] rigorously obtained all of these results and more via a “steepest descent method” for Riemann-Hilbert problems.
The solution of the defocusing cubic Schrödinger equation is based on inverse-scattering for the self-adjoint Zakharov-Shabat operator. To compute the direct scattering transform $R_0(\lambda) := \mathcal{S}[\psi_0](\lambda)$, one views the initial data $\psi_0$ as a given “potential” in the spectral problem

$$\frac{du}{dx} = \begin{bmatrix} -i\lambda & \psi_0(x) \\ \psi_0(x)^* & i\lambda \end{bmatrix} u.$$

the corresponding “reflection coefficient” $R_0(\lambda) := \mathcal{S}[\psi_0](\lambda)$ is then defined for $\lambda \in \mathbb{R}$ as a ratio of Wronskians of certain Jost solutions. For our initial data:

$$R_0(\lambda) := \frac{Be^{-2iL\lambda} \sin(2L\sqrt{\lambda^2 - |B|^2})}{\sqrt{\lambda^2 - |B|^2} \cos(2L\sqrt{\lambda^2 - |B|^2}) - i\lambda \sin(2L\sqrt{\lambda^2 - |B|^2})}.$$
Note that as $B \to 0$, $R_0(\lambda)^* = \mathcal{J}[\psi_0](\lambda)^*$ becomes the Fourier transform $\hat{\psi}_0(\lambda) = \mathcal{F}[\psi_0](\lambda)$:

Also, for $\delta > 0$ small enough there is a constant $K_\delta > 0$ such that the estimate $|R_0(\lambda)| \leq K_\delta|\lambda|^{-1}$ holds for $|\text{Im}(\lambda)| \leq \delta$ (i.e., $R_0(\lambda) = O(\lambda^{-1})$ uniformly in thin horizontal strips containing $\mathbb{R}$).
The reflection coefficient evolves with time exactly as the (conjugate) Fourier
transform does under the linear equation, namely \( R(\lambda, t) = R_0(\lambda)e^{2i\lambda^2 t} \). The
inverse transform \( \psi(x, t) = \mathcal{S}^{-1}[R(\cdot, t)](x, t) \) is calculated via the solution of
a Riemann-Hilbert problem. Given \( R_0(\lambda) \), form the “jump matrix” \( V(\lambda; x, t) \),
\( \lambda \in \mathbb{R} \):

\[
V(\lambda; x, t) = \begin{bmatrix}
1 - |R_0(\lambda)|^2 & -R_0(\lambda)^*e^{-2i(x\lambda+t\lambda^2)} \\
R_0(\lambda)e^{2i(x\lambda+t\lambda^2)} & 1
\end{bmatrix}.
\]

Then seek a \( 2 \times 2 \) matrix function \( M(\lambda; x, t) \) satisfying:

1. \( M(\lambda; x, t) \) is analytic for \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).
2. \( M(\lambda; x, t) \to \mathbb{I} \) as \( \lambda \to \infty \).
3. For \( \lambda \in \mathbb{R} \), \( M_+(\lambda; x, t) = M_-(\lambda; x, t)V(\lambda; x, t) \).

Here \( M_{\pm}(\lambda; x, t) := \lim_{\epsilon \downarrow 0} M(\lambda \pm i\epsilon; x, t) \) for \( \lambda \in \mathbb{R} \). Then

\[
\psi(x, t) = 2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda; x, t).
\]

Due to the slow decay of \( R_0(\lambda) \), this limit does not exist for \( x = \pm L \) and
\( t = 0 \), but it does otherwise.
Setting $x = ct$, the jump matrix for the Riemann-Hilbert problem is

$$V(\lambda; ct, t) = \begin{bmatrix} 1 - |R_0(\lambda)|^2 & -R_0(\lambda)^* e^{itI_c(\lambda)} \\ R_0(\lambda) e^{-itI_c(\lambda)} & 1 \end{bmatrix}$$

where $I_c(\lambda) := -2(c\lambda + \lambda^2)$ is the same phase function as in the linear theory. Notice that both exponentials $e^{itI_c(\lambda)}$ and $e^{-itI_c(\lambda)}$ appear. For analytic $R_0(\lambda)$ it is possible to replace the real axis $\mathbb{R}$ by another suitable contour $C$. But recalling the sign chart of $\text{Re}(iI_c(\lambda))$:

<table>
<thead>
<tr>
<th>$\text{Re}(iI_c(\lambda)) &lt; 0$</th>
<th>$\text{Re}(iI_c(\lambda)) &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_c$</td>
<td></td>
</tr>
</tbody>
</table>

we see that there is no single contour $C$ along which both exponentials are decaying as $t \to +\infty$. 
Observe that one can factor the jump matrix two ways:

\[
    V(\lambda; ct, t) = \begin{bmatrix}
        1 & -R_0(\lambda) e^{itI c(\lambda)} \\
        0 & 1
    \end{bmatrix} \begin{bmatrix}
        1 & 0 \\
        R_0(\lambda) e^{-itI c(\lambda)} & 1
    \end{bmatrix},
\]

and (noting that \(|R_0(\lambda)| < 1\) holds for all \(\lambda \in \mathbb{R}\))

\[
    V(\lambda; ct, t) = \begin{bmatrix}
        1 & 0 \\
        \frac{R_0(\lambda) e^{-itI c(\lambda)}}{1 - |R_0(\lambda)|^2} & 1
    \end{bmatrix} (1 - |R_0(\lambda)|^2)^{\sigma_3} \begin{bmatrix}
        1 & -\frac{R_0(\lambda) e^{itI c(\lambda)}}{1 - |R_0(\lambda)|^2} \\
        0 & 1
    \end{bmatrix}.
\]

Here \(a^{\sigma_3} := \text{diag}(a, a^{-1})\). These factorizations algebraically separate the different exponential factors \(e^{\pm itI c(z)}\). A key point is then that different factors of the jump matrix can be placed onto different contours in the complex plane!
Solving a Riemann-Hilbert problem is not the same as evaluating an integral. However, there is an analogue of Cauchy’s Integral Theorem that allows jump contours to be moved or subdivided in the complex plane.

This procedure is sometimes called *deformation* of a Riemann-Hilbert problem. It can be used to give a noncommutative analogue of classical steepest descent.
We take advantage of the two factorizations of $V(\lambda; ct, t)$ by explicitly transforming $M(\lambda; ct, t)$ into $N(\lambda; c, t)$ as follows:

$$N := M \begin{bmatrix} 1 & \frac{R_0(\lambda^*) e^{itI_c(\lambda)}}{1 - R_0(\lambda)R_0(\lambda^*)^*} \\ 0 & 1 \end{bmatrix}$$

(Elsewhere in the complex plane, $N := M$.)
Long-Time Asymptotics for Dispersive Waves

Deift-Zhou steepest descent technique. Exploiting the factorizations. Steepest descent.

$N(\lambda; c, t)$ is analytic for $\lambda \in \mathbb{C} \setminus \Sigma_N$, $N(\lambda; c, t) \to I$ as $\lambda \to \infty$, and for each $\lambda \in \Sigma_N$ the jump condition $N_-(\lambda; c, t) = N_+(\lambda; c, t)V_N(\lambda; c, t)$ holds:

$$V_N = \begin{bmatrix} 1 & -\frac{R_0(\lambda^*)e^{itI_c(\lambda)}}{1 - R_0(\lambda)R_0(\lambda^*)^*} \\ 0 & \frac{1}{1 - R_0(\lambda)R_0(\lambda^*)^*} \end{bmatrix}$$

$$V_N = \begin{bmatrix} \frac{1}{R_0(\lambda)}e^{-itI_c(\lambda)} & 0 \\ 1 & 1 \end{bmatrix}$$
Long-Time Asymptotics for Dispersive Waves

Deift-Zhou steepest descent technique. Parametrix construction.

The point is that \( V_N \to I \) rapidly as \( t \to +\infty \) except near \((-\infty, \lambda_c]\):

- Along the ray \((-\infty, \lambda_c]\) the jump matrix \( V_N \) is diagonal and independent of \( t \), but is not close to \( I \).
- On the cross \((\times)\) of \( \Sigma_N \), convergence \( V_N \to I \) is not uniform.

Both of these can be addressed using an explicit approximation to \( N(\lambda; c, t) \) called a parametrix, denoted \( \dot{N}(\lambda; c, t) \) and having two parts:

- An outer parametrix is designed to approximate \( N \) by simply replacing its jump matrix by its pointwise limit as \( t \to +\infty \). We denote the outer parametrix by \( \dot{N}^{(\text{out})}(\lambda; c) \). The outer parametrix will have a discontinuity at \( \lambda_c \), and it also poorly approximates \( N \) near \( \lambda_c \) because it doesn’t take into account jumps on the cross \((\times)\), which are not near-identity.

- An inner parametrix is a Riemann-Hilbert analogue of a boundary layer. It is designed to approximate \( N \) near \( \lambda_c \) and to match well with \( \dot{N}^{(\text{out})} \) far from \( \lambda_c \). We denote the inner parametrix by \( \dot{N}^{(\text{in})}(\lambda; c, t) \).
Long-Time Asymptotics for Dispersive Waves
Deift-Zhou steepest descent technique. Outer parametrix.

Since the limiting jump matrix $V_N(\lambda; c, t) = (1 - |R_0(\lambda)|^2) \sigma_3$ on $(-\infty, \lambda_c]$ is diagonal, we may seek the outer parametrix as a diagonal matrix too. Thus $\hat{N}^{(\text{out})}(\lambda; c) = f_c(\lambda) \sigma_3$, where $f_c(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus (-\infty, \lambda_c]$, $f_c(\lambda) \to 1$ as $\lambda \to \infty$ and

$$f_{c+}(\lambda) = f_{c-}(\lambda)(1 - |R_0(\lambda)|^2), \quad \lambda < \lambda_c.$$  

Taking logarithms using the fact that $|R_0(\lambda)| < 1$ gives the additive jump condition

$$\log(f_{c+}(\lambda)) - \log(f_{c-}(\lambda)) = \ln(1 - |R_0(\lambda)|^2), \quad \lambda < \lambda_c.$$  

We need $\log(f_c(\lambda)) \to 0$ as $\lambda \to \infty$, so this jump condition is uniquely solved by the Plemelj formula. Thus:

$$f_c(\lambda) := \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\lambda_c} \frac{\ln(1 - |R_0(s)|^2)}{s - \lambda} \, ds \right).$$
$\mathcal{N}^{(\text{out})}$ has a singularity at $\lambda_c$. Indeed, $L(\lambda) = \log(1 - R_0(\lambda)R_0(\lambda^*)^*)$ is an analytic function agreeing with $\ln(1 - |R_0(\lambda)|^2)$ for $\lambda \in \mathbb{R}$, and then

$$\int_{-\infty}^{\lambda_c} \frac{\ln(1 - |R_0(s)|^2)}{s - \lambda} ds = L(\lambda) \log(\lambda - \lambda_c) - L(\lambda) \log(1 + \lambda - \lambda_c)$$

$$+ \int_{-\infty}^{\lambda - 1} \frac{L(s)}{s - \lambda} ds + \int_{\lambda - 1}^{\lambda_c} \frac{L(s) - L(\lambda)}{s - \lambda} ds, \quad \lambda \in \mathbb{C} \setminus (-\infty, \lambda_c].$$

All but the first term are analytic functions of $\lambda$ at $\lambda_c$, so we get

$$\int_{-\infty}^{\lambda_c} \frac{\ln(1 - |R_0(s)|^2)}{s - \lambda} ds = L(\lambda_c) \log(\lambda - \lambda_c) + A_c + O((\lambda - \lambda_c) \log(\lambda - \lambda_c)),$$

$$A_c := \int_{-\infty}^{\lambda - 1} \frac{L(s)}{s - \lambda_c} ds + \int_{\lambda - 1}^{\lambda_c} \frac{L(s) - L(\lambda_c)}{s - \lambda_c} ds.$$

Hence, $f_c(\lambda) = e^{\frac{A_c}{2\pi i}} (\lambda - \lambda_c)^{\frac{L(\lambda_c)}{2\pi i}} (1 + O((\lambda - \lambda_c) \log(\lambda - \lambda_c))).$
To construct the inner parametrix, approximate the jump matrix $V_N$ close to $\lambda_c$. Let $D_c(t)$ be a disk of radius $t^{-p}$ centered at $\lambda_c$ where $0 < p < \frac{1}{2}$. Since $R_0(\lambda) = R_0(\lambda_c) + O(t^{-p})$ and $R_0(\lambda^*)^* = R_0(\lambda_c)^* + O(t^{-p})$ hold uniformly in $D_c(t)$, the jump matrix satisfies $V_N = \dot{V}(I + O(t^{-p}))$ where $\dot{V}$ is as indicated:

\[
\begin{bmatrix}
1 & -m e^{i\theta} e^{-i\zeta^2} \\
0 & \frac{1 - m^2}{1 - m^2}
\end{bmatrix}
\]

\[
(1 - m^2)\sigma_3
\]

\[
\begin{bmatrix}
1 \\
me^{-i\theta} e^{i\zeta^2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
me^{-i\theta} e^{i\zeta^2} & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -m e^{i\theta} e^{-i\zeta^2} \\
0 & 1
\end{bmatrix}
\]

Here, $\zeta = \sqrt{2t}(\lambda - \lambda_c)$, $m = |R_0(\lambda_c)| < 1$, and $\theta = \frac{1}{2}tc^2 - \arg(R_0(\lambda_c))$. 
We seek $\dot{N}^{(in)}(\lambda; c, t)$ as a matrix that satisfies the model jump condition

$\dot{N}^{(in)}_+ = \dot{N}^{(in)}_- \dot{V}$ on the part of $\Sigma_N$ within $D_c(t)$, is otherwise analytic within $D_c(t)$, and that matches well onto $\dot{N}^{(out)}(\lambda; c)$ in the sense that

$$\sup_{\lambda \in \partial D_p(t)} \left\| \dot{N}^{(in)}(\lambda; c, t) \dot{N}^{(out)}(\lambda; c)^{-1} - \mathbb{I} \right\| \ll 1, \quad t \to +\infty.$$ 

Since, in terms of the “inner” variable $\zeta = \sqrt{2t}(\lambda - \lambda_c)$ we have

$$\dot{N}^{(out)}(\lambda; c) = e^{i\left(\frac{L(\lambda_c)}{4\pi} \ln(2t) - \frac{A_c}{2\pi}\right)\sigma_3 \frac{L(\lambda_c)}{2\pi i} \sigma_3 (1 + O(t^{-p} \ln(t)))\sigma_3}, \quad \lambda \in \partial D_c(t),$$

and noting that $\lambda \in \partial D_c(t)$ means $\zeta \to \infty$ as $t \to +\infty$, we will set

$$\dot{N}^{(in)}(\lambda; c, t) = e^{i\left(\frac{L(\lambda_c)}{4\pi} \ln(2t) - \frac{A_c}{2\pi}\right)\sigma_3} \mathbf{P}(\zeta).$$
Long-Time Asymptotics for Dispersive Waves
Deift-Zhou steepest descent technique. Inner parametrix.

Then, we define \( P(\zeta) \) as the matrix analytic in the whole \( \zeta \)-plane except for \( \arg(-\zeta) = 0, \pm \pi/4, \pm 3\pi/4 \) along which rays it satisfies \( P_+ = P_- \hat{V} \) and that is normalized so that \( P(\zeta) \zeta^{-\frac{L(\lambda_c)}{2\pi i}} \sigma_3 \to \mathbb{I} \) as \( \zeta \to \infty \). We can remove the phase factors \( e^{\pm i\theta} \) from \( \hat{V} \) by setting \( P(\zeta) = e^{i\frac{\theta}{2} \sigma_3} Q(\zeta) e^{-i\frac{\theta}{2} \sigma_3} \) and observe that \( Q(\zeta) \) satisfies \( Q_+ = Q_- W \), where \( W \) is as indicated:

\[
\begin{bmatrix}
1 & -me^{-i\zeta^2} \\
0 & 1 - m^2 \\
(1 - m^2) \sigma_3 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
me^{i\zeta^2} & 1 \\
1 & -me^{-i\zeta^2} \\
0 & 1
\end{bmatrix}
\]

Also \( Q(\zeta) \zeta^{-\frac{L(\lambda_c)}{2\pi i}} \sigma_3 = Q(\zeta) \zeta^{-\frac{\ln(1-m^2)}{2\pi i}} \sigma_3 \to \mathbb{I} \) as \( \zeta \to \infty \).
The Riemann-Hilbert conditions defining $Q(\zeta)$ are essentially those of the isomonodromy problem identified by Its. $Q(\zeta)$ is expressed in terms of parabolic cylinder functions as follows. Set $U(\zeta) := Q(\zeta)e^{-i\zeta^2\sigma_3/2}$. Then the jump matrix $C$ for $U$ is constant on each ray:

$$
\begin{bmatrix}
1 & -\frac{m}{1-m^2} \\
0 & 1
\end{bmatrix}
\begin{cases}
(1-m^2)\sigma_3 & \text{for } \zeta \to +\infty \\
\frac{1}{m} & \text{for } \zeta \to -\infty \\
\frac{1}{1-m^2} & \text{for } \zeta \to 0
\end{cases}
\begin{bmatrix}
1 & 0 \\
m & 1
\end{bmatrix}
\begin{bmatrix}
1 & -m \\
0 & 1
\end{bmatrix}
\begin{cases}
1 & \text{for } \zeta \to +\infty \\
0 & \text{for } \zeta \to 0
\end{cases}
\begin{cases}
1 & \text{for } \zeta \to -\infty \\
m & \text{for } \zeta \to 0
\end{cases}
$$

Differentiating $U_+ = U_- C$ shows that $U'(\zeta)$ satisfies exactly the same jump conditions. Therefore $U'(\zeta)U(\zeta)^{-1}$ is an entire function.
Supposing that the normalization of $Q$ holds in the stronger sense that
\[
Q(\zeta) = (\mathbb{I} + \zeta^{-1}Q^{(1)} + \cdots)\zeta^{\frac{\ln(1-m^2)}{2\pi i}}\sigma_3, \quad \zeta \to \infty
\]
with the series differentiable term-by-term, we see that the entire function $U'(\zeta)U(\zeta)^{-1}$ satisfies
\[
U'(\zeta)U(\zeta)^{-1} = Q'(\zeta)Q(\zeta)^{-1} - i\zeta Q(\zeta)\sigma_3 Q(\zeta)^{-1}
\]
\[
= -i\zeta \sigma_3 + i[\sigma_3, Q^{(1)}] + O(\zeta^{-1}), \quad \zeta \to \infty.
\]
By Liouville’s Theorem then,
\[
U'(\zeta)U(\zeta)^{-1} = -i\zeta \sigma_3 + i[\sigma_3, Q^{(1)}] = \begin{bmatrix}
-i\zeta & 2iQ_{12}^{(1)} \\
-2iQ_{21}^{(1)} & i\zeta
\end{bmatrix}.
\]
In other words,
\[
\frac{dU}{d\zeta} = \begin{bmatrix}
-i\zeta & 2iQ_{12}^{(1)} \\
-2iQ_{21}^{(1)} & i\zeta
\end{bmatrix}U.
\]
It can be shown that $Q(z^*)^* = \sigma_1 Q(z)\sigma_1$, which implies that $Q_{21}^{(1)} = Q_{12}^{(1)*}$, so if $\beta := 2iQ_{12}^{(1)}$, the differential equations for $U$ read

$$\frac{dU}{d\zeta} = \begin{bmatrix} -i\zeta & \beta \\ \beta^* & i\zeta \end{bmatrix} U.$$

Now go to dlmf.nist.gov...
The procedure is now the following. Consider $\beta \in \mathbb{C}$ as a parameter to be determined in terms of $0 \leq m < 1$.

1. For arbitrary $\beta \in \mathbb{C}$, express the general solution matrix $U$ as $U_0(\zeta)K$ where $U_0$ is a fundamental matrix built from the parabolic cylinder function $U(a, z)$ and $K$ is a constant matrix.

2. In each sector of analyticity of $U$, determine the corresponding constant matrix $K$ such that $U(\zeta)e^{i\zeta^2/2}\zeta^{-\ln(1-m^2)/2\pi i} \to \mathbb{I}$ as $\zeta \to \infty$. This uses known asymptotic formulas for $U(a, z)$ when $z$ is large. This step reveals that $|\beta|^2 = -\frac{1}{\pi} \ln(1 - m^2)$.

3. Now impose the constant jump conditions on $U(\zeta)$ across the sector boundaries. Using known connection formulas relating $U(a, z)$, $U(a, -z)$, and $U(-a, \pm iz)$, one finds that

$$\arg(\beta) = \frac{\pi}{4} + \frac{\ln(2)}{2\pi} \ln(1 - m^2) - \arg \left( \Gamma \left( -\frac{\ln(1 - m^2)}{2\pi i} \right) \right).$$
Thus $\beta = \beta(m)$ is completely determined, and so is $U(\zeta)$. For example, if $|\arg(\zeta)| < \pi/4$, then setting $a := \frac{1}{2}(1 + i|\beta|^2)$, we have

$$U(\zeta) = \frac{1}{\sqrt{1 - m^2}} \begin{bmatrix} U(-a, \sqrt{2}e^{i\pi/4}\zeta) & -\frac{e^{i\pi/4}\beta}{\sqrt{2}} U(a, \sqrt{2}e^{-i\pi/4}\zeta) \\ -\frac{e^{-i\pi/4}\beta^*}{\sqrt{2}} U(a^*, \sqrt{2}e^{i\pi/4}\zeta) & U(-a^*, \sqrt{2}e^{-i\pi/4}\zeta) \end{bmatrix} e^{-\frac{1}{4\pi i} \ln(2) \ln(1-m^2) \sigma_3}.$$ 

Also there is a constant $k = k(c)$ such that in every sector,

$$U(\zeta)e^{i\zeta^2/2}e^{-\frac{\ln(1-m^2)}{2\pi i} \sigma_3} = 1 + \frac{1}{2i\zeta} \begin{bmatrix} 0 & \beta \\ -\beta^* & 0 \end{bmatrix} + \frac{k\sigma_3}{\zeta^2} + O(\zeta^{-3}), \quad \zeta \to \infty.$$ 

Finally, the inner parametrix is given explicitly by:

$$\hat{N}^{(in)}(\lambda; c, t) := e^{i\left(\frac{1}{4\pi} \ln(1-m^2) \ln(2t) - \frac{A_c}{2\pi}\right)\sigma_3} e^{i\theta\sigma_3/2} U(\zeta) e^{i\zeta^2/2} e^{-i\theta\sigma_3/2}$$

where $\zeta = \sqrt{2t}(\lambda - \lambda_c)$, $m = |R_0(\lambda_c)| < 1$, and $\theta = \frac{1}{2}tc^2 - \arg(R_0(\lambda_c))$. 
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Deift-Zhou steepest descent technique. Error analysis.

Given an exponent $0 < p < \frac{1}{2}$, the global parametrix for $\mathbf{N}(\lambda; c, t)$ is

$$\dot{\mathbf{N}}(\lambda; c, t) := \begin{cases} 
\dot{\mathbf{N}}^{\text{(out)}}(\lambda; c), & |\lambda - \lambda_c| > t^{-p} \\
\dot{\mathbf{N}}^{\text{(in)}}(\lambda; c, t), & |\lambda - \lambda_c| < t^{-p}.
\end{cases}$$

To determine the accuracy of modeling the unknown matrix $\mathbf{N}(\lambda; c, t)$ with the explicit parametrix $\dot{\mathbf{N}}(\lambda; c, t)$, consider the error

$$\mathbf{E}(\lambda; c, t) := \mathbf{N}(\lambda; c, t)\dot{\mathbf{N}}(\lambda; c, t)^{-1},$$

which should be close to the identity matrix throughout the complex $\lambda$-plane. This matrix is analytic for $\lambda \in \mathbb{C} \setminus \Sigma_{\mathbf{E}}$ as pictured:
For each non-intersection point $\lambda \in \Sigma_E$, we have a jump condition (sign convention: + on the left) $E_+ = E_- Z$, where the jump matrix $Z$ is known:

$$Z(\lambda) := \begin{cases} \dot{N}_- (\lambda) (V_N (\lambda) \dot{V} (\lambda)^{-1}) \dot{N}_- (\lambda)^{-1}, & \lambda \in \Sigma_E \setminus \partial D_c (t) \\ \dot{N}^{(in)} (\lambda) \dot{N}^{(out)} (\lambda)^{-1}, & \lambda \in \partial D_c (t). \end{cases}$$

Here, $V_N$ (resp., $\dot{V}$) is the jump matrix for $N$ (resp., $\dot{N}$). Also, since both $N(\lambda; c, t)$ and $\dot{N}(\lambda; c, t)$ tend to $I$ as $\lambda \to \infty$, the same is true for $E(\lambda; c, t)$.

Therefore, $E(\lambda; c, t)$ satisfies the conditions of its own Riemann-Hilbert problem, with known data. Also, because $f_c (\lambda) \to 1$ as $\lambda \to \infty$, and because $M(\lambda; x = ct, t) = N(\lambda; c, t) = E(\lambda; c, t) \dot{N}(\lambda; c, t) = E(\lambda; c, t) f_c (\lambda)^{\sigma_3}$ holds for $|\text{Im}(\lambda)|$ sufficiently large,

$$\psi(ct, t) := 2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda; ct, t) = 2i \lim_{\lambda \to \infty} \lambda E_{12}(\lambda; c, t).$$
We can actually solve for $E(\lambda; c, t)$ by converting the Riemann-Hilbert problem into a singular integral equation. Writing the jump condition as

$$E_+(\lambda) - E_-(\lambda) = E_-(\lambda)(Z(\lambda) - I), \quad \lambda \in \Sigma_E,$$

and using the Plemelj formula with $E(\lambda) \to I$ as $\lambda \to \infty$ gives

$$E(\lambda; c, t) = I + \frac{1}{2\pi i} \int_{\Sigma_E} \frac{E_-(s; c, t)(Z(s) - I)}{s - \lambda} ds, \quad \lambda \in \mathbb{C} \setminus \Sigma_E.$$

Letting $\lambda$ tend to $\Sigma_E$ from the right ($-$) side gives a closed singular integral equation for $F(\lambda) := E_-(\lambda; c, t) - I$:

$$F(\lambda) - S[F](\lambda) = G(\lambda), \quad \lambda \in \Sigma_E, \quad G(\lambda) := C_-[Z - I](\lambda),$$

where

$$C_-[H](\lambda) := \frac{1}{2\pi i} \int_{\Sigma_E} \frac{H(s)}{s - \lambda} ds \quad \text{and} \quad S[F](\lambda) := C_-[F(Z - I)](\lambda).$$
It is known that for each $t > 0$, the linear operator $C_-$ is bounded on $L^2(\Sigma_E)$. Moreover, it can be shown that, although $\Sigma_E$ depends on $t$ through the radius $t^{-p}$ of $D_c(t)$, the operator norm $\|C_-\|_2$ is bounded independent of $t$. From the definition of $S$ we then get the following estimate:

$$\|S\|_2 \leq \|C_-\|_2 \|Z - I\|_\infty.$$ 

It follows that, given $t > 0$,

- if $Z - I \in L^2(\Sigma_E) \cap L^\infty(\Sigma_E)$ and
- if $\|Z - I\|_\infty$ is sufficiently small,

then the singular integral equation has a unique solution by convergent Neumann series:

$$F(\lambda) = G(\lambda) + S[G](\lambda) + S[S[G]](\lambda) + \cdots.$$ 

If $Z - I$ tends to zero as $t \to \infty$ in suitable norms, then this is also an asymptotic expansion of $F(\lambda)$ and hence of $E(\lambda; ct, t)$ in the limit $t \to \infty$. 
From the Neumann series we get the estimate
\[ \|F\|_2 \leq \frac{\|G\|_2}{1 - \|S\|_2} \leq \frac{\|C_-\|_2 \|Z - I\|_2}{1 - \|C_-\|_2 \|Z - I\|_\infty}. \]

Since \( E_- = I + F \) on \( \Sigma_E \), we get a formula for the error \( E(\lambda) \):
\[ E(\lambda) = I + \frac{1}{2\pi i} \int_{\Sigma_E} \frac{Z(s) - I}{s - \lambda} \, ds + \frac{1}{2\pi i} \int_{\Sigma_E} \frac{F(s)(Z(s) - I)}{s - \lambda} \, ds, \quad \lambda \in \mathbb{C} \setminus \Sigma_E. \]

Therefore, multiplying by \( 2i\lambda \) and letting \( \lambda \to \infty \) in the 12-component,
\[ \psi(ct, t) = -\frac{1}{\pi} \int_{\Sigma_E} Z_{12}(s) \, ds - \frac{1}{\pi} \int_{\Sigma_E} [F(s)(Z(s) - I)]_{12} \, ds. \]

By Cauchy-Schwarz, the second integral is \( O(\|F\|_2 \|Z - I\|_2) \). To continue further, we have to take a closer look at \( Z(\lambda) - I \) for \( \lambda \in \Sigma_E \).
Let the contour $\Sigma_E$ be the disjoint union of
- $\Sigma_E^{(\text{out})} := \{ \lambda \in \Sigma_E, |\lambda - \lambda_c| > t^{-p} \}$,
- $\Sigma_E^{(\text{in})} := \{ \lambda \in \Sigma_E, |\lambda - \lambda_c| < t^{-p} \}$, and
- $\Sigma_E^{(\circ)} := \{ \lambda \in \Sigma_E, |\lambda - \lambda_c| = t^{-p} \}$.

Analysis of $Z - \mathbb{I}$ restricted to $\Sigma_E^{(\text{out})}$.

For $\lambda \in \Sigma_E^{(\text{out})}$, we have $Z - \mathbb{I} = \hat{N}_{\circ}^{(\text{out})} (V_N - \mathbb{I}) \hat{N}_{\circ}^{(\text{out})} - 1$, and the outer parametrix $\hat{N}_{\circ}^{(\text{out})}$ and its inverse are independent of $t$ and uniformly bounded in the plane.

Moreover, because $|\lambda - \lambda_c|$ is large compared to $t^{-1/2}$, the exponential factors $e^{\pm it I_c(\lambda)}$ can do their work, and therefore $(V_N - \mathbb{I})|_{\Sigma_E^{(\text{out})}}$ and hence also $(Z - \mathbb{I})|_{\Sigma_E^{(\text{out})}}$ is exponentially small in $L^1$, $L^2$, and $L^\infty$. 
Analysis of $Z - \mathbb{I}$ restricted to $\Sigma^{(\text{in})}_E$.

For $\lambda \in \Sigma^{(\text{in})}_E$, we have $Z - \mathbb{I} = \dot{N}^{(\text{in})}_-(V_N \dot{V}^{-1} - \mathbb{I})^{-1} \dot{N}^{(\text{in})}_-$, and the inner parametrix $\dot{N}^{(\text{in})}$ and its inverse are uniformly bounded in $D_c(t)$ independently of $t$.

Moreover, by Taylor expansion, we have $(V_N \dot{V}^{-1} - \mathbb{I})|_{\Sigma^{(\text{in})}_E} = O(t^{-p})$ in $L^\infty$, and hence the same holds for $(Z - \mathbb{I})|_{\Sigma^{(\text{in})}_E}$. Because the arc length of $\Sigma^{(\text{in})}_E$ is proportional to $t^{-p}$, it follows that $(Z - \mathbb{I})|_{\Sigma^{(\text{in})}_E} = O(t^{-2p})$ in $L^1$ and that $(Z - \mathbb{I})|_{\Sigma^{(\text{in})}_E} = O(t^{-3p/2})$ in $L^2$. 

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Analysis of $Z - \mathbb{I}$ restricted to $\Sigma^{(o)}_E$.

Finally, for $\lambda \in \Sigma^{(o)}_E$, we have $Z - \mathbb{I} = \hat{N}^{(in)} \hat{N}^{(out)} - 1 - \mathbb{I}$. For $\lambda \in \Sigma^{(o)}_E$,

$$Z(\lambda) = \hat{N}^{(in)}(\lambda) \hat{N}^{(out)}(\lambda)^{-1} = e^{i \varphi(c, t) \sigma_3} \left( \mathbb{I} + \frac{1}{2i\zeta} \begin{bmatrix} 0 & \beta \\ -\beta^* & 0 \end{bmatrix} + \frac{k \sigma_3}{\zeta^2} + O(\zeta^{-3}) \right) \left( 1 + O(t^{-p} \ln(t)) \right) \sigma_3 e^{-i \varphi(c, t) \sigma_3},$$

where the real phase $\varphi(c, t)$ is defined as

$$\varphi(c, t) := \frac{1}{4\pi} \ln(1 - m^2) \ln(2t) - \frac{A_c}{2\pi} + \frac{\theta}{2}.$$

Because $|\zeta| = \sqrt{2}t^{1/2-p}$ on $\Sigma^{(o)}_E$, $(Z - \mathbb{I})|_{\Sigma^{(o)}_E} = O(t^{p-1/2}) + O(t^{-p} \ln(t))$ in $L^\infty$. Since the arc length of $\Sigma^{(o)}_E$ is proportional to $t^{-p}$ it follows that $(Z - \mathbb{I})|_{\Sigma^{(o)}_E} = O(t^{(p-1)/2}) + O(t^{-3p/2} \ln(t))$ in $L^2$. 
By the triangle inequality,
\[
Z - \mathbb{I} = \begin{cases} 
O(t^{p-1/2}) + O(t^{-p} \ln(t)) & \text{in } L^\infty(\Sigma_E) \\
O(t^{(p-1)/2}) + O(t^{-3p/2} \ln(t)) & \text{in } L^2(\Sigma_E).
\end{cases}
\]
For both norms, restricting \(p\) further by assuming that \(\frac{1}{4} < p < \frac{1}{2}\) makes the algebraic estimate dominant. These imply that the Neumann series for \(F\) converges for \(t\) large enough, and that \(F = O(t^{(p-1)/2})\) in \(L^2(\Sigma_E)\). Therefore,

\[
\psi(ct, t) = -\frac{1}{\pi} \int_{\Sigma_E} Z_{12}(s) \, ds + O(t^{p-1}).
\]

Furthermore, by the \(L^1\) estimates of \(Z - \mathbb{I}\) on \(\Sigma_E^{(\text{out})}\) and \(\Sigma_E^{(\text{in})}\),

\[
\psi(ct, t) = -\frac{1}{\pi} \int_{\Sigma_E^{(\text{out})}} Z_{12}(s) \, ds + O(t^{-2p}) + O(t^{p-1}).
\]

The error terms here are optimally balanced for \(p = 1/3\) at \(O(t^{-2/3})\).
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It remains to integrate $Z_{12}(\lambda)$ over the small circle $\Sigma^{(o)}_E$ of radius $t^{-p}$. Recall the uniform approximation

$$Z_{12}(\lambda) = e^{2i\varphi(c,t)} \left( \frac{\beta}{2i\zeta} + O(\zeta^{-3}) \right) \left( 1 + O(t^{-p} \ln(t)) \right), \quad \lambda \in \Sigma^{(o)}_E.$$  

Taking $p = 1/3$ and noting that $\lambda \in \Sigma^{(o)}_E$ means $|\zeta| = \sqrt{2}t^{1/6}$,

$$Z_{12}(\lambda) = \frac{e^{2i\varphi(c,t)} \beta}{2i\zeta} + O(t^{-1/2} \ln(t)), \quad \lambda \in \Sigma^{(o)}_E.$$  

Since the arc length of $\Sigma^{(o)}_E$ is proportional to $t^{-1/3}$,

$$-\frac{1}{\pi} \int_{\Sigma^{(o)}_E} Z_{12}(s) \, ds = -\frac{e^{2i\varphi(c,t)} \beta}{2\pi i} \int_{\Sigma^{(o)}_E} \frac{ds}{\sqrt{2}t(s - \lambda_c)} + O(t^{-5/6} \ln(t))$$

$$= \frac{e^{2i\varphi(c,t)} \beta}{\sqrt{2}t} + O(t^{-5/6} \ln(t)).$$
Finally, we can write down an asymptotic formula for $\psi(ct, t)$:

$$
\psi(ct, t) = \frac{e^{2i\varphi(c,t)}\beta}{\sqrt{2t}} + O(t^{-2/3}), \quad t \to +\infty.
$$

The leading term can be written in the Segur-Ablowitz form

$$
\frac{e^{2i\varphi(c,t)}\beta}{\sqrt{2t}} = t^{-1/2} \alpha(c) e^{itc^2/2} e^{-i\nu(c)\ln(t)}, \quad \nu(c) := -\frac{1}{2\pi} \ln(1 - |R_0(\lambda_c)|^2),
$$

where

$$
|\alpha(c)| := \sqrt{-\frac{\ln(1 - |R_0(\lambda_c)|^2)}{2\pi}},
$$

$$
\arg(\alpha(c)) := -\arg(R_0(\lambda_c)) + \frac{\pi}{4} - \arg\left(\Gamma\left(-\frac{\ln(1 - |R_0(\lambda_c)|^2)}{2\pi i}\right)\right)
$$

$$
+ \frac{\ln(2)}{\pi} \ln(1 - |R_0(\lambda_c)|^2) - \frac{A_c}{\pi}.
$$

Note: $\alpha(c) \sim R_0(\lambda_c)^* e^{-i\pi/4}/\sqrt{2\pi} \sim \mathscr{F}[\psi_0](\lambda_c) e^{-i\pi/4}/\sqrt{2\pi}$ as $B \to 0$. 

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Further generalizations.

Some notes:

- The same method applies, with the same asymptotic formula in terms of $R_0(\lambda_c)$, whenever $R_0(\lambda)$ is analytic in a thin strip about the real axis. But, in general it is difficult to calculate $R_0(\lambda_c)$ explicitly.

- $\mathcal{S}$ is like $\mathcal{F}$ in that decay (smoothness) of the initial condition implies corresponding smoothness (decay) of the reflection coefficient $R_0(\lambda)$. Without rapid decay of $\psi_0(x)$, we cannot assume analyticity of $R_0(\lambda)$, and the technique requires modification:
  - rational approximation of $R_0(\lambda)$ — highly technical. See [1].
  - $\tilde{\partial}$ steepest descent method — conceptually simpler. See [2].
References


