Sources of Nonuniformity

Sequence of matrix functions :

$$\mathbf{M}(\lambda) \to \mathbf{N}(\lambda) \to \tilde{\mathbf{N}}(\lambda) \to \mathbf{O}(\lambda) \to \tilde{\mathbf{O}}(\lambda)$$

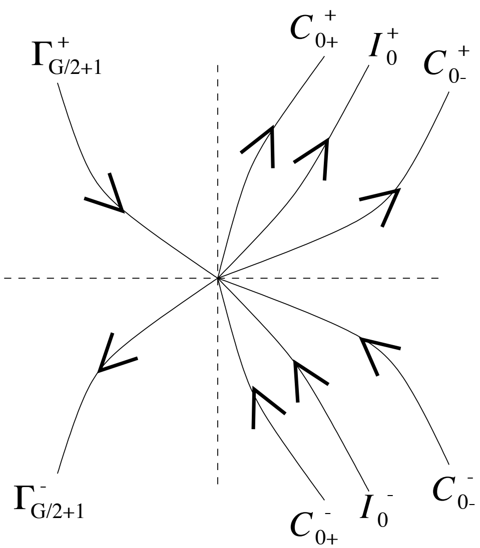
Ad-hoc steps:

1. Continuum Limit of Jump Matrix: $\mathbf{N}(\lambda) \to \tilde{\mathbf{N}}(\lambda)$

2. Pointwise Asymptotics of Jump Matrix: $\mathbf{O}(\lambda) \to \tilde{\mathbf{O}}(\lambda)$

Both of these break down near $\lambda = 0$.

Inner Asymptotics near $\lambda = 0$

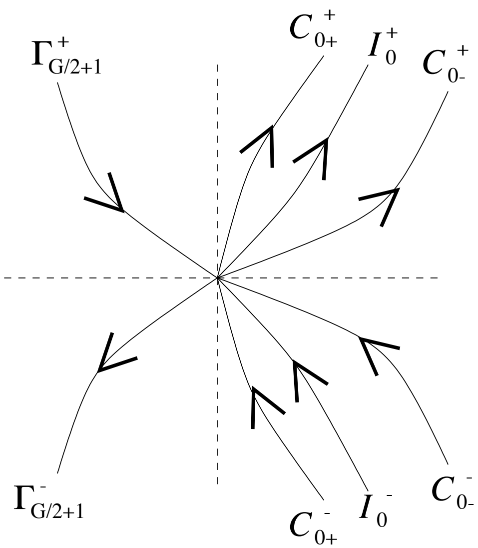


Discrepancy of approximation of $\mathbf{N}(\lambda)$ by $\hat{\mathbf{N}}_{\mathsf{out}}(\lambda)$ is the quotient $\mathbf{N}(\lambda)\hat{\mathbf{N}}_{\mathsf{out}}(\lambda)^{-1}$.

Convenient to introduce a conjugation by an explicit, holomorphic matrix $\mathbf{C}(\lambda)$ and look at

$$\mathbf{F}(\lambda) := \mathbf{C}(\lambda)^{-1}\mathbf{N}(\lambda)\hat{\mathbf{N}}_{\mathsf{out}}(\lambda)^{-1}\mathbf{C}(\lambda)\,.$$

Exact jump relation: $\mathbf{F}_+(\lambda) = \mathbf{F}_-(\lambda)\mathbf{v}_\mathbf{F}(\lambda)$ with



$$\mathbf{v}_\mathbf{F}(\lambda) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -ie^{\delta/\hbar}e^{(\tilde{\phi}(\lambda)-\tilde{\phi}(0))/\hbar}(1-d(\lambda)) & 1 \end{bmatrix}, & \lambda \in \Gamma^+_{G/2+1} \\ \begin{bmatrix} 1 & ie^{-i(\theta(\lambda)-\theta(0))/\hbar} \\ 0 & 1 \end{bmatrix}, & \lambda \in C^+_{0+} \\ \begin{bmatrix} 1 & 0 \\ ie^{i(\theta(\lambda)-\theta(0))/\hbar} & 1 \end{bmatrix}, & \lambda \in C^+_{0-} \\ \begin{bmatrix} 1-d(\lambda) & ie^{-i(\theta(\lambda)-\theta(0))/\hbar}d(\lambda) \\ ie^{i(\theta(\lambda)-\theta(0))/\hbar}d(\lambda) & 1+d(\lambda) \end{bmatrix}, & \lambda \in I^+_0 \end{cases}$$

and $\mathbf{v}_\mathbf{F}(\lambda) = \sigma_2 \mathbf{v}_\mathbf{F}(\lambda^*)^* \sigma_2$.

$$\tilde{\phi}(\lambda) := \int_0^{iA} L^0_\eta(\lambda)\rho^0(\eta)\,d\eta + \int_{-iA}^0 L^0_\eta(\lambda)\rho^0(\eta^*)^*\,d\eta + 2i\lambda x + 2i\lambda^2 t + i\pi\int_\lambda^{iA}\rho^0(\eta)\,d\eta - g_+(\lambda) - g_-(\lambda)\,.$$

$$d(\lambda) = 1 - \left[\prod_{n=0}^{N-1}\frac{\lambda - \lambda^{\mathsf{WKB}*}_{\hbar_N,n}}{\lambda - \lambda^{\mathsf{WKB}}_{\hbar_N,n}}\right]\exp\left(-\frac{1}{\hbar_N}\left[\int_0^{iA} L^0_\eta(\lambda)\rho^0(\eta)\,d\eta + \int_{-iA}^0 L^0_\eta(\lambda)\rho^0(\eta^*)^*\,d\eta\right]\right)$$

$$\theta(\lambda) = -\pi\int_\lambda^0 \rho(\eta)\,d\eta\,, \quad L^0_\eta(\lambda) := \log(-i(\lambda-\eta)) + i\pi/2\,, \quad e^{\delta/\hbar} = 1 \quad \text{for} \quad \hbar = \hbar_N\,.$$

Approximating the Jump Matrix Near the Origin

1. Approximate $\theta(\lambda) - \theta(0)$ and $\tilde{\phi}(\lambda) - \tilde{\phi}(0)$ near the origin with the first term in their Taylor series.

2. Approximate $d(\lambda)$ uniformly away from the imaginary axis using the "ladder of eigenvalues".

Express asymptotics in terms of a rescaled variable $\zeta = -i\rho^0(0)\lambda/\hbar$.

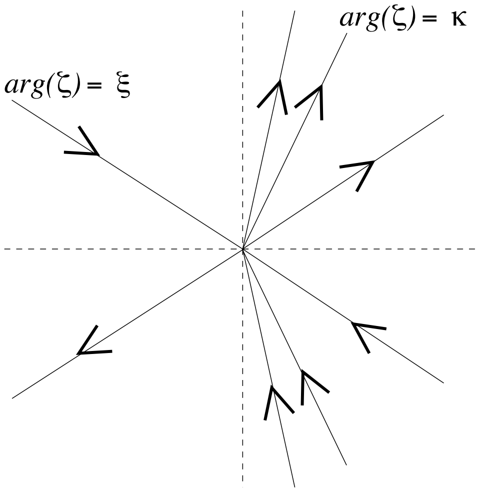
Ultimately: we'll use the approximation we are developing in place of $\hat{\mathbf{N}}_{\text{out}}(\lambda)$ in a neighborhood of $\lambda = 0$ of radius $\hbar^\epsilon$ with $1/2 < \epsilon < 1$. Later: error is optimized with $\epsilon = 2/3$.

Define $u = u(x,t)$ and $v = v(x,t)$:

$$e^{(\tilde{\phi}(\lambda)-\tilde{\phi}(0))/\hbar} = e^{u\zeta}e^{O(\lambda^2/\hbar)} \qquad e^{\pm i(\theta(\lambda)-\theta(0))/\hbar} = e^{\pm iv\zeta}e^{O(\lambda^2/\hbar)}$$

Take all contours except $I_0^\pm$ to be straight rays (w.l.o.g.). Then *replace* $I_0^\pm$ by their tangent rays.

The Model Riemann-Hilbert Problem Near the Origin



$$\mathbf{v}_{\hat{\mathbf{F}}}(\zeta) := \begin{cases} \begin{bmatrix} 1 & 0 \\ -i(1-h(\zeta))e^{(u-2\pi)\zeta} & 1 \end{bmatrix}, & \arg(\zeta) = \xi\,, \\ \begin{bmatrix} 1 & ie^{-iv\zeta} \\ 0 & 1 \end{bmatrix}, & \arg(\zeta) = \kappa/2 + \pi/4\,, \\ \begin{bmatrix} 1 & 0 \\ ie^{iv\zeta} & 1 \end{bmatrix}, & \arg(\zeta) = \kappa/2\,, \\ \begin{bmatrix} 1-h(\zeta) & ih(\zeta)e^{-iv\zeta} \\ ih(\zeta)e^{iv\zeta} & 1+h(\zeta) \end{bmatrix}, & \arg(\zeta) = \kappa\,. \end{cases}$$

$$h(\zeta) = 1 - \frac{\Gamma(1/2+i\zeta)}{\Gamma(1/2-i\zeta)}(-i\zeta)^{-2i\zeta}e^{(2i+\pi)\zeta} \quad \text{and} \quad \mathbf{v}_{\hat{\mathbf{F}}}(\zeta) = \sigma_2 \mathbf{v}_{\hat{\mathbf{F}}}(\zeta^*)^* \sigma_2$$

Riemann-Hilbert Problem: Find $\hat{\mathbf{F}}(\zeta)$ analytic in $\mathbb{C} \setminus \Sigma_{\hat{\mathbf{F}}}$ with

1. $\hat{\mathbf{F}}(\zeta) \to \mathbb{I}$ as $\zeta \to \infty$
2. Continuous boundary values satisfying $\hat{\mathbf{F}}_+(\zeta) = \hat{\mathbf{F}}_-(\zeta)\mathbf{v}_{\hat{\mathbf{F}}}(\zeta)$.

Solvability of the Model

Fact: There is a unique solution of this Riemann-Hilbert problem with the additional property that

$$\widehat{\mathbf{F}}(\zeta) - \mathbb{I} = O(|\zeta|^{-1})\,.$$

Proof based on general theory of RHPs with jump matrices in Hölder spaces. Correspondence with systems of singular integral equations of Fredholm type.

Normalization matrix $\mathbb{I}$ at infinity — an inhomogeneity. The Fredholm alternative applies because

1. $\mathbf{v}_{\widehat{\mathbf{F}}}(\zeta)$ is Hölder continuous (but not Lipschitz) on each ray.
2. $\mathbf{v}_{\widehat{\mathbf{F}}}(\zeta) - \mathbb{I} = O(|\zeta|^{-1})$ as $\zeta \to \infty$.
3. Cyclic relation holds at the self-intersection point.

Unique solvability follows upon ruling out homogeneous solutions. We exploit the Schwartz reflection symmetry of $\mathbf{v}_{\widehat{\mathbf{F}}}(\zeta)$ to do this.

Decay estimate for the solution at infinity: vanishing of the sum of the moments of $\mathbf{v}_{\widehat{\mathbf{F}}}(\zeta) - \mathbb{I}$ over all rays.

Local Parametrix Near the Origin

The relation

$$\mathbf{N}(\lambda) = \mathbf{C}(\lambda)\mathbf{F}(\lambda)\mathbf{C}(\lambda)^{-1}\hat{\mathbf{N}}_{\mathsf{out}}(\lambda)$$

holds exactly.

From $\widehat{\mathbf{F}}(\zeta(\lambda))$, build an approximation $\widehat{\mathbf{G}}(\lambda)$ for $\widehat{\mathbf{F}}(\zeta(\lambda))$ by "un-straightening" $I_0^{\pm}$ for $|\lambda| < \hbar^{2/3}$.

Since $\widehat{\mathbf{G}}(\lambda)$ is expected to be a good approximation to $\mathbf{F}(\lambda)$, we build an improved approximation to $\mathbf{N}(\lambda)$ valid near $\lambda = 0$ by setting

$$\hat{\mathbf{N}}_{\mathsf{origin}}(\lambda) := \mathbf{C}(\lambda)\hat{\mathbf{G}}(\lambda)\mathbf{C}(\lambda)^{-1}\hat{\mathbf{N}}_{\mathsf{out}}(\lambda)$$

Variational Theory of the Complex Phase

Green's function for upper half-plane: $G(\lambda;\eta) := \log\left|\frac{\lambda-\eta^*}{\lambda-\eta}\right|$

External field:

$$\varphi(\lambda) := -\int G(\lambda;\eta)\,d\mu^0(\eta) - \Re\left(i\pi\sigma\int_\lambda^{iA}\rho^0(\eta)\,d\eta + 2iJ(\lambda x+\lambda^2 t)\right)$$

$d\mu^0 =$ nonnegative asymptotic WKB eigenvalue measure on $[0, iA]$

Energy functional: $E[d\mu] := \frac{1}{2}\int d\mu(\lambda)\int G(\lambda;\eta)\,d\mu(\eta) + \int \varphi(\lambda)\,d\mu(\lambda)$

Equilibrium Property

Theorem 1 *Let $\rho(\eta)$ be an admissible density function on the oriented loop contour C surrounding $[0, iA]$. Then*

$$E[-\rho(\eta)\,d\eta] = \inf_{d\mu} E[d\mu]$$

where the infimum is taken over all nonnegative Borel measures supported on C and having finite mass and finite Green's energy.

Idea of proof: let $d\Delta(\eta) := d\mu(\eta) + \rho(\eta)\,d\eta$. Then

$$E[d\mu] - E[-\rho(\eta)\,d\eta] = \frac{1}{2}\int d\Delta(\lambda) \int G(\lambda;\eta)\,d\Delta(\eta) + \int \Re(\tilde{\phi}(\lambda))\,d\Delta(\lambda)$$

1. First term is nonnegative because positive and negative parts of $d\Delta$ have finite mass and Green's energy.

2. Second term is nonnegative because:

   (a) $\Re(\tilde{\phi}(\lambda)) \equiv 0$ when λ is in the support of $\rho(\eta)\,d\eta$

   (b) $\Re(\tilde{\phi}(\lambda)) \leq 0$ when λ is outside the support of $\rho(\eta)\,d\eta$, and consequently where $d\Delta(\lambda) = d\mu(\lambda) \geq 0$.

S-Property

Theorem 2 *Let $\rho(\eta)$ be an admissible density function on an oriented loop contour C surrounding $[0, iA]$. For each $\kappa(\eta)$ analytic in the support of $-\rho(\eta)\, d\eta$ on C and satisfying $\kappa(0) = 0$ and for each sufficiently small ϵ let $d\mu_\epsilon^\kappa$ be the pull-back of the measure $-\rho(\eta)\, d\eta$ under the near-identity map*

$$\nu_\epsilon^\kappa : \eta \to \eta + \epsilon\kappa(\eta)\,.$$

Then

$$\frac{d}{d\epsilon} E[d\mu_\epsilon^\kappa]\Big|_{\epsilon=0} = 0\,.$$

Idea of proof: Using the pull-back property,

$$E[d\mu_\epsilon^\kappa] = \frac{1}{2}\int_d \mu_0^\kappa(\lambda) \int G(\nu_\epsilon^\kappa(\lambda); \nu_\epsilon^\kappa(\eta))\, d\mu_0^\kappa(\eta) + \int \varphi(\nu_\epsilon^\kappa(\lambda)\, d\mu_0^\kappa(\lambda)$$

where $d\mu_0^\kappa(\eta) = -\rho(\eta)\, d\eta$. Find that

$$\frac{d}{d\epsilon} E[d\mu_\epsilon^\kappa]\Big|_{\epsilon=0} = -\int \Re\left[\kappa(\lambda)\frac{d}{d\lambda}\tilde{\phi}(\lambda)\right]\, d\mu_0^\kappa(\lambda)$$

which vanishes because $\tilde{\phi}(\lambda)$ is a constant function along the contour in the support of $-\rho(\eta)\, d\eta$.

Nature of the Critical Point. Max-Min Problem.

Energy functional is:

1. Minimized by $-\rho(\eta)\,d\eta$ over measures supported on the fixed contour C.
2. Stationary with respect to deformations of C with the measure “held fixed”.

Can assign an equilibrium energy $E_{\text{min}}[C]$ to arbitrary loop contours C. But property 2 not necessarily equivalent to $E_{\text{min}}[C]$ being stationary with respect to deformations of C.

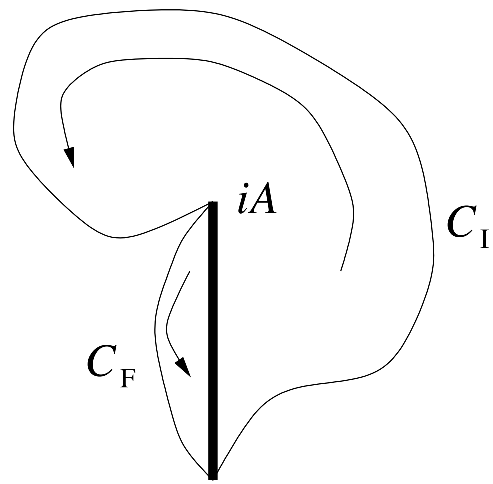
Want to pose a “max-min” problem: For each contour C find the equilibrium energy $E_{\text{min}}[C]$ over all positive Borel measures $d\mu$ supported on C. Then pick C so as to maximize $E_{\text{min}}[C]$.

Generalization of the method of Lax and Levermore for zero dispersion Korteweg-de Vries. But, energy problem does not play as central a role in our analysis. Further understanding is required.

We hope: study of the variational problem will provide existence, uniqueness, and regularity (finite number of bands and gaps) for the complex phase. A “hunting licence”. Maybe an upper bound on the number of bands.

Seeking the Complex Phase by Ansatz

Suppose that C passes through iA and all bands lie on one half, C_I:



Guess a number of bands and gaps on C_I ($2G + 2$ complex endpoints, in conjugate pairs, with G even), and seek scalar $F(\lambda)$ analytic in $\mathbb{C} \setminus (C_I \cup C_I^*)$ satisfying

$$F(\lambda^*) = -F(\lambda)^* \quad \text{and} \quad F(\lambda) = O(1/\lambda) \quad \text{as} \quad \lambda \to \infty$$

and on C_I,

$$\begin{aligned} F_+(\lambda) &+ F_-(\lambda) &= -4iJ(x + 2\lambda t)\,, &\quad \lambda \text{ in a band} \\ F_+(\lambda) &- F_-(\lambda) &= -2\pi i \rho^0(\lambda)\,, &\quad \lambda \text{ in a gap} \end{aligned}$$

Then get a "candidate density function" via

$$\rho(\eta) = \rho^0(\eta) + \frac{1}{2\pi i}(F_+(\eta) - F_-(\eta))\,.$$

1. Consistency of this procedure imposes $G+1$ real "moment conditions" on the endpoints.

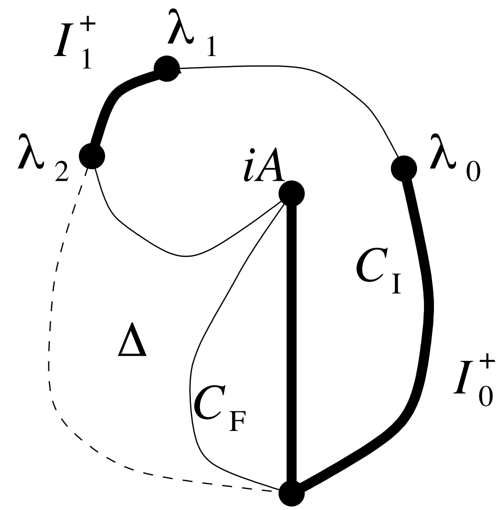
2. Procedure guarantees only that $\rho(\eta) \equiv 0$ in the gaps and $\tilde{\phi}(\lambda)$ is constant in the bands.

3. $G/2$ additional real "vanishing conditions" may be imposed to ensure that $\tilde{\phi}(\lambda)$ is purely imaginary in the bands.

4. $G/2+1$ additional real "measure reality conditions" are required if $\rho(\eta)\,d\eta$ is to be real in the bands (*i.e.* for $\theta(\eta)$ to be real).

Total of $2G+2$ real conditions on $2G+2$ independent real unknowns.

Once $F(\lambda)$ is found, pull contour C away from iA:



Finally verify:

1. That there are actually contours connecting the band endpoints along which $\rho(\eta)\,d\eta$ is real,

2. That the inequalities $\Re(\tilde{\phi}(\lambda)) < 0$ in gaps and $\rho(\eta)\,d\eta < 0$ in bands are satisfied.

These conditions would select the genus G as a function of x and t.

Genus Zero

Only one complex endpoint $\lambda_0 = a_0 + ib_0 \in \mathbb{C}_+$ and two real conditions:

$$M_0 = -2J\pi(x + 2a_0 t) + 2\Re\left(\int_{\lambda_0}^{iA} \frac{i\pi\rho^0(\eta)}{R(\eta)}\, d\eta\right) = 0$$

$$R_0 = -Jtb_0^2 + \Im\left(\int_{\lambda_0}^{iA} \rho^0(\eta)\frac{\partial R}{\partial \eta}(\eta)\, d\eta\right) = 0$$

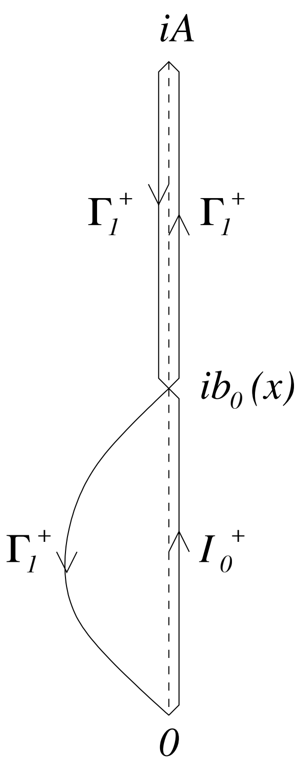
Here $R(\eta)^2 = (\eta - \lambda_0)(\eta - \lambda_0^*)$, branch cut along the bands $I_0^{\pm}$ and $R(\eta) \sim -\eta$ as $\eta \to \infty$.

The $G = 0$ Ansatz for $t = 0$

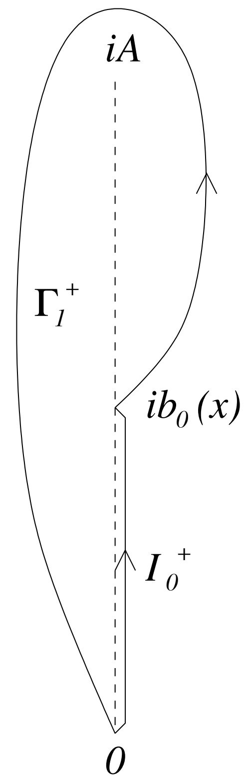
Using formula for $\rho^0(\eta)$ in terms of $A(x)$ one finds that for $t = 0$

$$a_0(x) = 0 \quad \text{and} \quad b_0(x) = A(x)$$

follow from the conditions $M_0 = R_0 = 0$.



Deform, respecting $\Re(\tilde{\phi}(\lambda)) < 0$ in the gaps:



Small Time Results

Theorem 3 *Let $A(x)$ be real-analytic, even, and monotone decreasing in $|x|$. Then for each fixed $x \neq 0$, a genus zero ansatz satisfies all properties of a complex phase function for t sufficiently small.*

Idea of proof:

1. Use properties of $A(x)$ to compute the Jacobian of the transformation $(\lambda_0, \lambda_0^*) \to (M_0, R_0)$ and show it is nonzero for $t = 0$. This shows persistence of the endpoints for t small.

2. Appeal to a fixed-point argument showing the persistence of the contour band and gaps for t small. Show that the ansatz can be rigged so that the band moves *away* from $[0, iA]$.

Theorem 4 *For sufficiently small t, the semiclassical soliton ensemble $\psi(x,t)$ associated with $A(x)$ is pointwise $\hbar^{1/3}$-close to $\tilde{\psi}(x,t) := A(x,t)e^{iS(x,t)}$ where $A(x,t)$ and $S(x,t)$ are the unique analytic solutions of the genus zero elliptic modulation equations with initial data $A(x,0) = A(x)$ and $S(x,0) = 0$.*

Finite t with $A(x) = A\text{sech}(x)$

About the endpoint $\lambda_0 = a_0 + ib_0$:

- Reality condition $R_0 = 0$ consistent only if $\sigma J t \geq 0$, and then

$$a_0^2 = t^2 b_0^4 \frac{A^2 - b_0^2 + t^2 b_0^4}{A^2 + t^2 b_0^4}$$

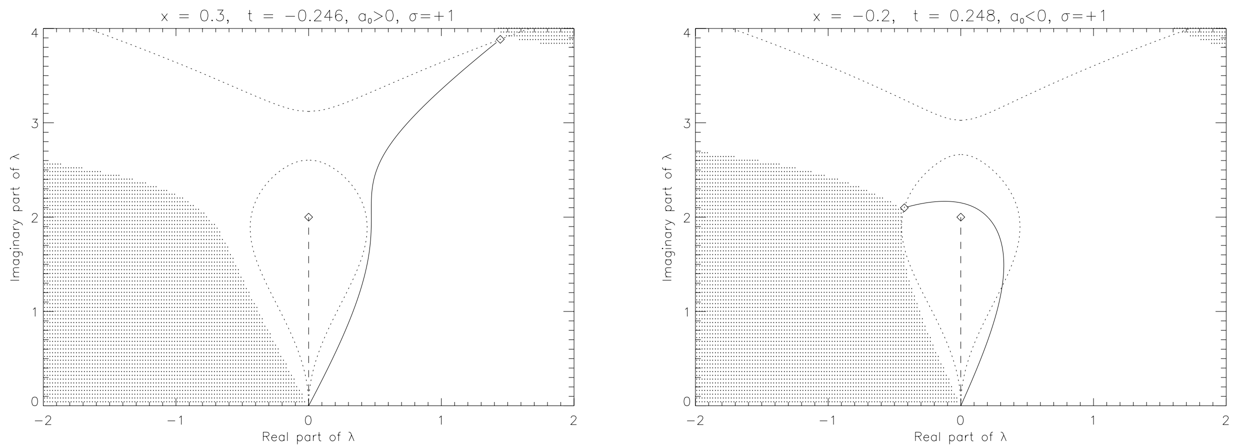
- Two solutions for the endpoint $\lambda_0(x,t)$, in left/right half-planes. One at infinity when $t = 0$.

Computer-assisted exploration. For given (x,t), chose one of the two possible endpoints. Then construct the candidate density $\rho(\eta)$ and

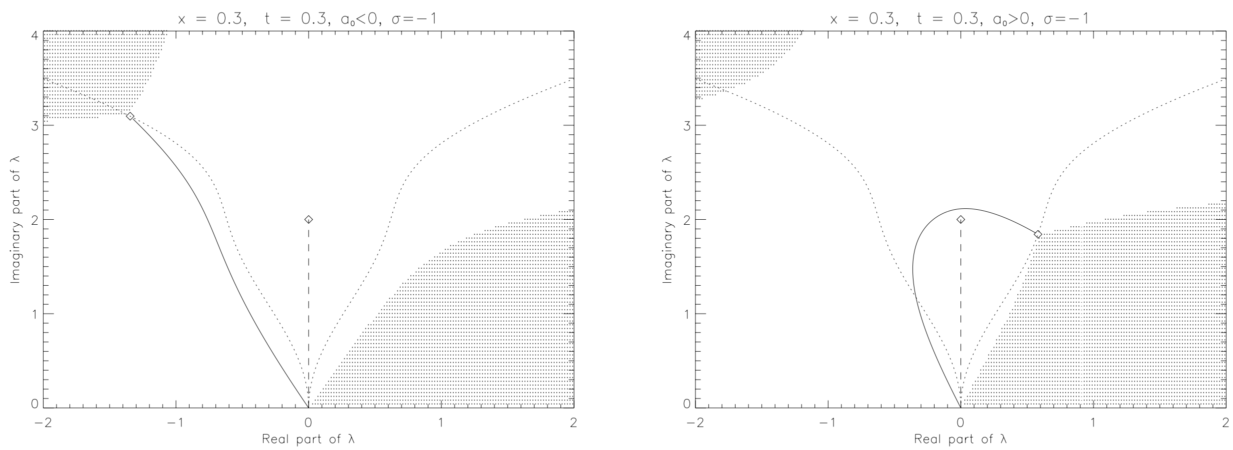
1. Numerically follow the orbit $\rho(\eta)\,d\eta < 0$ from the origin and see whether it makes it to λ_0 safely. This determines whether the band I_0^+ can exist.

2. If I_0^+ exists, numerically construct $\Re(\tilde{\phi}(\lambda))$ and see where it is negative. Determine whether the contour C can be closed around $[0, iA]$ in such a region.

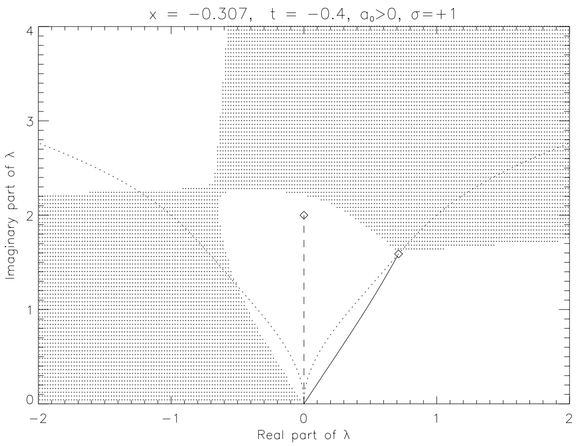
Comparing the two possible endpoints before breaktime:

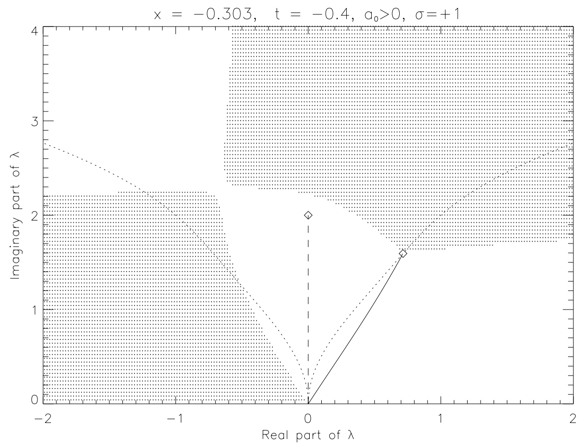


And after breaktime:

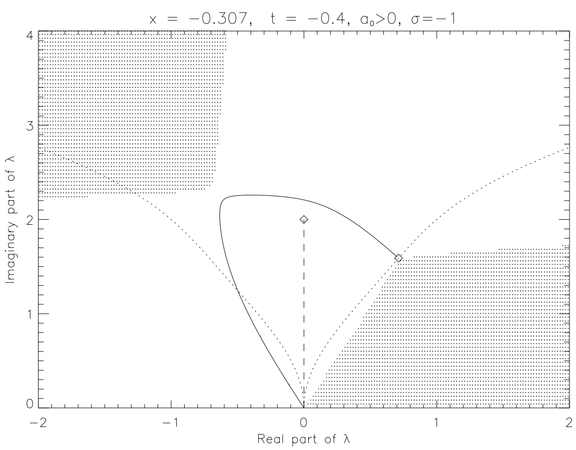


Breakdown of the ansatz: Failure of inequality in the gaps.

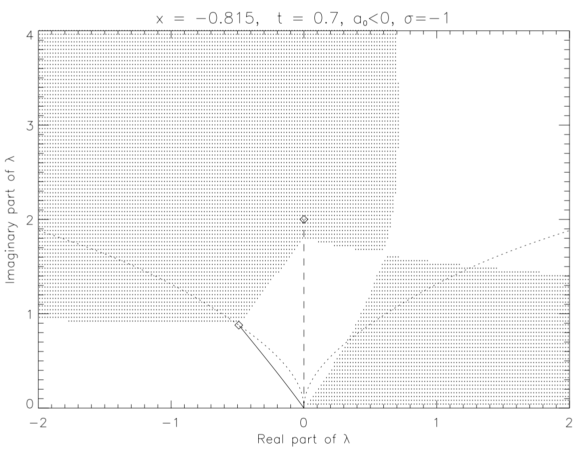


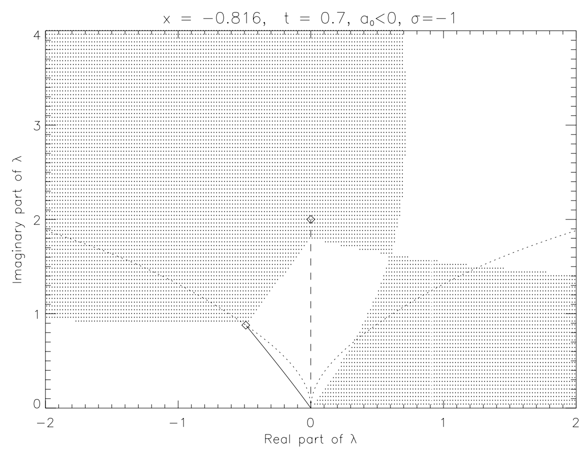


"Dual" ansatz: reverse roles of bands and gaps!

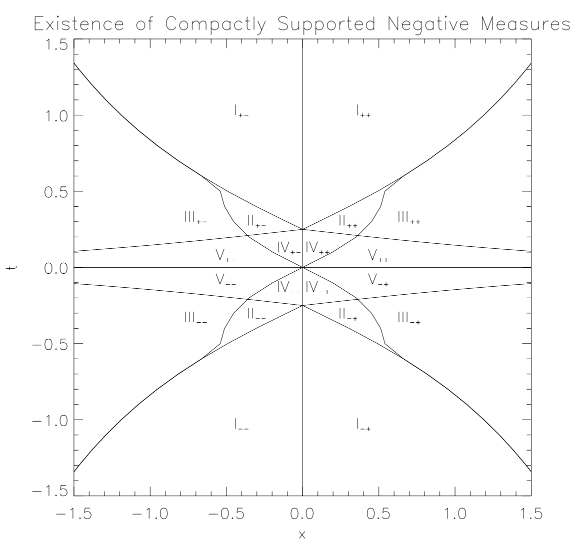


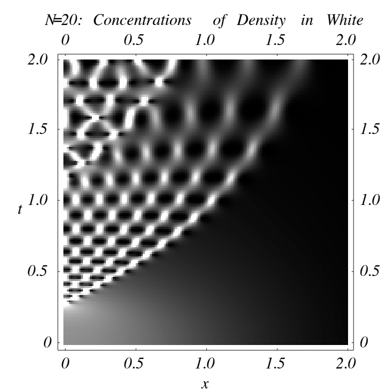
Another example of inequality failure in the gaps. No dual ansatz.





Complete scan of the (x, t)-plane:





Modes of Failure of the Ansatz. Phase Transition.

The ansatz can fail at some (x, t) in several ways:

1. The region admitting a gap contour can "pinch off".

2. A complex zero of $\rho(\eta)$ can move onto a band.

3. A band can strike the interval $[0, iA]$.

4. The endpoint functions can fail to be analytic.

Apparently the ansatz can be chosen so that case 1 is the mode of failure.

Theorem 5 *If the genus zero ansatz fails at a point $(x_{\text{crit}}, t_{\text{crit}})$ due to the pinching off of a gap at a point $\hat{\lambda}$ (not in the shadow of I_0^+) then for $|x|-|x_{\text{crit}}| <$ 0 and small enough in magnitude, a genus two ansatz suffices to generate a complex phase function.*