Sources of Nonuniformity

Sequence of matrix functions:

\[ M(\lambda) \rightarrow N(\lambda) \rightarrow \tilde{N}(\lambda) \rightarrow O(\lambda) \rightarrow \tilde{O}(\lambda) \]

Ad-hoc steps:

1. Continuum Limit of Jump Matrix: \( N(\lambda) \rightarrow \tilde{N}(\lambda) \)

2. Pointwise Asymptotics of Jump Matrix: \( O(\lambda) \rightarrow \tilde{O}(\lambda) \)

Both of these break down near \( \lambda = 0 \).
Inner Asymptotics near $\lambda = 0$

Discrepancy of approximation of $N(\lambda)$ by $\tilde{N}_{\text{out}}(\lambda)$ is the quotient $N(\lambda)\tilde{N}_{\text{out}}(\lambda)^{-1}$.

Convenient to introduce a conjugation by an explicit, holomorphic matrix $C(\lambda)$ and look at

$$F(\lambda) := C(\lambda)^{-1}N(\lambda)\tilde{N}_{\text{out}}(\lambda)^{-1}C(\lambda).$$
Exact jump relation: $F_+(\lambda) = F_-(\lambda)v_F(\lambda)$ with

$$v_F(\lambda) = \begin{cases} 
\begin{bmatrix} 1 & 0 \\
-ie^{\delta/h}(\phi(\lambda)-\phi(0))/h(1-d(\lambda)) & 1 
\end{bmatrix}, & \lambda \in \Gamma_{G/2}^+ \\
\begin{bmatrix} 1 & 0 \\
-ie^{-i(\theta(\lambda)-\theta(0))/h} & 1 
\end{bmatrix}, & \lambda \in C_{0+}^+ \\
\begin{bmatrix} 1 & 0 \\
i e^{-i(\theta(\lambda)-\theta(0))/h} & 1 
\end{bmatrix}, & \lambda \in C_{0-}^+ \\
\begin{bmatrix} 1-d(\lambda) & ie^{i(\theta(\lambda)-\theta(0))/h}d(\lambda) \\
i e^{i(\theta(\lambda)-\theta(0))/h}d(\lambda) & 1+d(\lambda) 
\end{bmatrix}, & \lambda \in I_0^+ 
\end{cases}$$

and $v_F(\lambda) = \sigma_2v_F(\lambda^\ast)^\ast\sigma_2$.

$$\phi(\lambda) := \int_0^{iA} L^0_\eta(\lambda)\rho^0(\eta) \, d\eta + \int_0^{-iA} L^0_\eta(\lambda)\rho^0(\eta^\ast) \, d\eta + 2i\lambda x + 2i\lambda^2 t + i\pi \int_0^{iA} \rho^0(\eta) \, d\eta - g_+(\lambda) - g_-(\lambda).$$

$$d(\lambda) = 1 - \prod_{n=0}^{N-1} \frac{\lambda - \lambda_{WKB}^{n+1}}{\lambda - \lambda_{WKB}^n} \exp \left(-\frac{1}{\hbar} \left[ \int_0^{iA} L^0_\eta(\lambda)\rho^0(\eta) \, d\eta + \int_{-iA}^0 L^0_\eta(\lambda)\rho^0(\eta^\ast) \, d\eta \right] \right)$$

$$\theta(\lambda) = -\pi \int_{1/\lambda} \rho(\eta) \, d\eta, \quad L^0_\eta(\lambda) := \log(-i(\lambda - \eta)) + i\pi/2, \quad e^{\delta/h} = 1 \quad \text{for} \quad \hbar = \hbar_N.$$
Approximating the Jump Matrix Near the Origin

1. Approximate $\theta(\lambda) - \theta(0)$ and $\tilde{\phi}(\lambda) - \tilde{\phi}(0)$ near the origin with the first term in their Taylor series.

2. Approximate $d(\lambda)$ uniformly away from the imaginary axis using the “ladder of eigenvalues”.

Express asymptotics in terms of a rescaled variable $\zeta = -i\rho^0(0)\lambda/h$.

Ultimately: we’ll use the approximation we are developing in place of $\tilde{N}_{\text{out}}(\lambda)$ in a neighborhood of $\lambda = 0$ of radius $h^\epsilon$ with $1/2 < \epsilon < 1$. Later: error is optimized with $\epsilon = 2/3$.

Define $u = u(x,t)$ and $v = v(x,t)$:

$$e^{(\tilde{\phi}(\lambda) - \tilde{\phi}(0))/h} = e^{u \zeta} e^{O(\lambda^2/h)} \quad e^{\pm i(\theta(\lambda) - \theta(0))/h} = e^{\pm i u \zeta} e^{O(\lambda^2/h)}$$

Take all contours except $I_0^{\pm}$ to be straight rays (w.l.o.g.). Then replace $I_0^{\pm}$ by their tangent rays.
The Model Riemann-Hilbert Problem Near the Origin

\[ v_F(\zeta) := \begin{cases} 
- i(1 - h(\zeta))e^{(u-2\pi)\zeta} & 0 \\
1 & 1 
\end{cases}, \quad \arg(\zeta) = \xi, \]

\[ \begin{bmatrix} 1 \\
0 \\
1 \\
0 
\end{bmatrix}, \quad \arg(\zeta) = \kappa/2 + \pi/4, \]

\[ \begin{bmatrix} 1 \\
0 \\
i e^{i\zeta} \\
1 
\end{bmatrix}, \quad \arg(\zeta) = \kappa/2, \]

\[ \begin{bmatrix} 1 - h(\zeta) & ih(\zeta)e^{-i\zeta} \\
0 & 1 + h(\zeta) 
\end{bmatrix}, \quad \arg(\zeta) = \kappa. \]

\[ h(\zeta) = 1 - \frac{\Gamma(1/2 + i\zeta)}{\Gamma(1/2 - i\zeta)}(-i\zeta)^{-2i\zeta}e^{2i(\pi + i)\zeta} \quad \text{and} \quad v_F(\zeta) = \sigma_2 v_F(\zeta^*)^* \sigma_2 \]

Riemann-Hilbert Problem: Find \( \hat{F}(\zeta) \) analytic in \( \mathbb{C} \setminus \Sigma_F \) with

1. \( \hat{F}(\zeta) \to \mathbb{I} \) as \( \zeta \to \infty \)

2. Continuous boundary values satisfying \( \hat{F}_+(\zeta) = \hat{F}_-(\zeta)v_F(\zeta) \).
Solvability of the Model

Fact: There is a unique solution of this Riemann-Hilbert problem with the additional property that

\[ \hat{F}(\zeta) - \mathbb{I} = O(|\zeta|^{-1}). \]

Proof based on general theory of RHPs with jump matrices in Hölder spaces. Correspondence with systems of singular integral equations of Fredholm type.

Normalization matrix \( \mathbb{I} \) at infinity — an inhomogeneity. The Fredholm alternative applies because

1. \( v_F(\zeta) \) is Hölder continuous (but not Lipschitz) on each ray.

2. \( v_F(\zeta) - \mathbb{I} = O(|\zeta|^{-1}) \) as \( \zeta \to \infty \).

3. Cyclic relation holds at the self-intersection point.

Unique solvability follows upon ruling out homogeneous solutions. We exploit the Schwartz reflection symmetry of \( v_F(\zeta) \) to do this.

Decay estimate for the solution at infinity: vanishing of the sum of the moments of \( v_F(\zeta) - \mathbb{I} \) over all rays.
Local Parametrix Near the Origin

The relation

\[ N(\lambda) = C(\lambda)F(\lambda)C(\lambda)^{-1}\tilde{N}_{\text{out}}(\lambda) \]

holds exactly.

From \( \hat{F}(\zeta(\lambda)) \), build an approximation \( \hat{G}(\lambda) \) for \( \hat{F}(\zeta(\lambda)) \) by “unstraightening” \( I_0^\pm \) for \(|\lambda| < \hat{\epsilon}^{2/3}\).

Since \( \hat{G}(\lambda) \) is expected to be a good approximation to \( F(\lambda) \), we build an improved approximation to \( N(\lambda) \) valid near \( \lambda = 0 \) by setting

\[ \hat{N}_{\text{origin}}(\lambda) := C(\lambda)\hat{G}(\lambda)C(\lambda)^{-1}\tilde{N}_{\text{out}}(\lambda) \]
Variational Theory of the Complex Phase

Green’s function for upper half-plane: \( G(\lambda; \eta) := \log \left| \frac{\lambda - \eta^*}{\lambda - \eta} \right| \)

External field:

\[
\varphi(\lambda) := - \int G(\lambda; \eta) \, d\mu^0(\eta) - \Re \left( i\pi \sigma \int_{\lambda}^{iA} \rho^0(\eta) \, d\eta + 2iJ(\lambda x + \lambda^2 t) \right)
\]

\( d\mu^0 = \) nonnegative asymptotic WKB eigenvalue measure on \([0, iA]\)

Energy functional: \( E[d\mu] := \frac{1}{2} \int d\mu(\lambda) \int G(\lambda; \eta) \, d\mu(\eta) + \int \varphi(\lambda) \, d\mu(\lambda) \)
Equilibrium Property

**Theorem 1** Let \( \rho(\eta) \) be an admissible density function on the oriented loop contour \( C \) surrounding \([0, iA]\). Then

\[
E[-\rho(\eta) \, d\eta] = \inf_{d\mu} E[d\mu]
\]

where the infimum is taken over all nonnegative Borel measures supported on \( C \) and having finite mass and finite Green’s energy.

Idea of proof: let \( d\Delta(\eta) := d\mu(\eta) + \rho(\eta) \, d\eta \). Then

\[
E[d\mu] - E[-\rho(\eta) \, d\eta] = \frac{1}{2} \int d\Delta(\lambda) \int G(\lambda; \eta) \, d\Delta(\eta) + \int \Re(\bar{\phi}(\lambda)) \, d\Delta(\lambda)
\]

1. First term is nonnegative because positive and negative parts of \( d\Delta \) have finite mass and Green’s energy.

2. Second term is nonnegative because:
   (a) \( \Re(\bar{\phi}(\lambda)) \equiv 0 \) when \( \lambda \) is in the support of \( \rho(\eta) \, d\eta \)
   (b) \( \Re(\bar{\phi}(\lambda)) \leq 0 \) when \( \lambda \) is outside the support of \( \rho(\eta) \, d\eta \), and consequently where \( d\Delta(\lambda) = d\mu(\lambda) \geq 0 \).
S-Property

**Theorem 2** Let \( \rho(\eta) \) be an admissible density function on an oriented loop contour \( \gamma \) surrounding \([0, iA]\). For each \( \kappa(\eta) \) analytic in the support of \( -\rho(\eta) \, d\eta \) on \( \gamma \) and satisfying \( \kappa(0) = 0 \) and for each sufficiently small \( \epsilon \) let \( d\mu^\kappa_\epsilon \) be the pull-back of the measure \( -\rho(\eta) \, d\eta \) under the near-identity map

\[
\nu^\kappa_\epsilon : \eta \rightarrow \eta + \epsilon \kappa(\eta).
\]

Then

\[
\frac{d}{d\epsilon} E[d\mu^\kappa_\epsilon] \bigg|_{\epsilon=0} = 0.
\]

Idea of proof: Using the pull-back property,

\[
E[d\mu^\kappa_\epsilon] = \frac{1}{2} \int d \mu^\kappa_0(\lambda) \int G(\nu^\kappa_\epsilon(\lambda); \nu^\kappa_\epsilon(\eta)) \, d\mu^\kappa_0(\eta) + \int \varphi(\nu^\kappa_\epsilon(\lambda)) \, d\mu^\kappa_0(\lambda)
\]

where \( d\mu^\kappa_0(\eta) = -\rho(\eta) \, d\eta \). Find that

\[
\frac{d}{d\epsilon} E[d\mu^\kappa_\epsilon] \bigg|_{\epsilon=0} = - \int \Re \left[ \kappa(\lambda) \frac{d}{d\lambda} \tilde{\phi}(\lambda) \right] \, d\mu^\kappa_0(\lambda)
\]

which vanishes because \( \tilde{\phi}(\lambda) \) is a constant function along the contour in the support of \( -\rho(\eta) \, d\eta \).
Nature of the Critical Point. Max-Min Problem.

Energy functional is:

1. Minimized by $-\rho(\eta)\,d\eta$ over measures supported on the fixed contour $C$.

2. Stationary with respect to deformations of $C$ with the measure “held fixed”.

Can assign an equilibrium energy $E_{\min}[C]$ to arbitrary loop contours $C$. But property 2 not necessarily equivalent to $E_{\min}[C]$ being stationary with respect to deformations of $C$.

Want to pose a “max-min” problem: For each contour $C$ find the equilibrium energy $E_{\min}[C]$ over all positive Borel measures $d\mu$ supported on $C$. Then pick $C$ so as to maximize $E_{\min}[C]$.

Generalization of the method of Lax and Levermore for zero dispersion Korteweg-de Vries. But, energy problem does not play as central a role in our analysis. Further understanding is required.

We hope: study of the variational problem will provide existence, uniqueness, and regularity (finite number of bands and gaps) for the complex phase. A “hunting licence”. Maybe an upper bound on the number of bands.
Seeking the Complex Phase by Ansatz

Suppose that $C$ passes through $iA$ and all bands lie on one half, $C_I$:

Guess a number of bands and gaps on $C_I$ ($2G + 2$ complex endpoints, in conjugate pairs, with $G$ even), and seek scalar $F(\lambda)$ analytic in $\mathbb{C} \setminus (C_I \cup C_I^*)$ satisfying

$$F(\lambda^*) = -F(\lambda)^* \quad \text{and} \quad F(\lambda) = O(1/\lambda) \quad \text{as} \quad \lambda \to \infty$$

and on $C_I$,

$$F_+(\lambda) + F_-(\lambda) = -4iJ(x + 2\lambda t), \quad \lambda \text{ in a band}$$

$$F_+(\lambda) - F_-(\lambda) = -2\pi i \rho^0(\lambda), \quad \lambda \text{ in a gap}$$
Then get a “candidate density function” via

\[ \rho(\eta) = \rho^0(\eta) + \frac{1}{2\pi i} (F_+(\eta) - F_-(\eta)) . \]

1. Consistency of this procedure imposes \( G + 1 \) real “moment conditions” on the endpoints.

2. Procedure guarantees only that \( \rho(\eta) \equiv 0 \) in the gaps and \( \tilde{\phi}(\lambda) \) is constant in the bands.

3. \( G/2 \) additional real “vanishing conditions” may be imposed to ensure that \( \tilde{\phi}(\lambda) \) is purely imaginary in the bands.

4. \( G/2 + 1 \) additional real “measure reality conditions” are required if \( \rho(\eta) \, d\eta \) is to be real in the bands (i.e. for \( \theta(\eta) \) to be real).

Total of \( 2G + 2 \) real conditions on \( 2G + 2 \) independent real unknowns.
Once $F(\lambda)$ is found, pull contour $C$ away from $iA$:

Finally verify:

1. That there are actually contours connecting the band endpoints along which $\rho(\eta) \, d\eta$ is real,

2. That the inequalities $\Re(\tilde{\phi}(\lambda)) < 0$ in gaps and $\rho(\eta) \, d\eta < 0$ in bands are satisfied.

These conditions would select the genus $G$ as a function of $x$ and $t$. 
Genus Zero

Only one complex endpoint $\lambda_0 = a_0 + ib_0 \in \mathbb{C}_+$ and two real conditions:

$$M_0 = -2J\pi(x + 2a_0 t) + 2\Re \left( \int_{\lambda_0}^{iA} \frac{i\pi \rho^0(\eta)}{R(\eta)} d\eta \right) = 0$$

$$R_0 = -Jtb_0^2 + \Im \left( \int_{\lambda_0}^{iA} \rho^0(\eta) \frac{\partial R}{\partial \eta}(\eta) d\eta \right) = 0$$

Here $R(\eta)^2 = (\eta - \lambda_0)(\eta - \lambda_0^*)$, branch cut along the bands $I_0^\pm$ and $R(\eta) \sim -\eta$ as $\eta \to \infty$. 
The \( G = 0 \) Ansatz for \( t = 0 \)

Using formula for \( \rho^0(\eta) \) in terms of \( A(x) \) one finds that for \( t = 0 \)

\[
a_0(x) = 0 \quad \text{and} \quad b_0(x) = A(x)
\]

follow from the conditions \( M_0 = R_0 = 0 \).

Deform, respecting \( \Re(\tilde{\phi}(\lambda)) < 0 \) in the gaps:
Small Time Results

**Theorem 3** Let $A(x)$ be real-analytic, even, and monotone decreasing in $|x|$. Then for each fixed $x \neq 0$, a genus zero ansatz satisfies all properties of a complex phase function for $t$ sufficiently small.

Idea of proof:

1. Use properties of $A(x)$ to compute the Jacobian of the transformation $(\lambda_0, \lambda_0) \rightarrow (M_0, R_0)$ and show it is nonzero for $t = 0$. This shows persistence of the endpoints for $t$ small.

2. Appeal to a fixed-point argument showing the persistence of the contour band and gaps for $t$ small. Show that the ansatz can be rigged so that the band moves away from $[0, iA]$.

**Theorem 4** For sufficiently small $t$, the semiclassical soliton ensemble $\psi(x,t)$ associated with $A(x)$ is pointwise $\hbar^{1/3}$-close to $\tilde{\psi}(x,t) := A(x,t)e^{iS(x,t)}$ where $A(x,t)$ and $S(x,t)$ are the unique analytic solutions of the genus zero elliptic modulation equations with initial data $A(x,0) = A(x)$ and $S(x,0) = 0$. 

17
Finite \( t \) with \( A(x) = \text{Asech}(x) \)

About the endpoint \( \lambda_0 = a_0 + ib_0 \):

- Reality condition \( R_0 = 0 \) consistent only if \( \sigma J t \geq 0 \), and then

\[
a_0^2 = t^2 b_0^4 \frac{A^2 - b_0^2 + t^2 b_0^4}{A^2 + t^2 b_0^4}
\]

- Two solutions for the endpoint \( \lambda_0(x, t) \), in left/right half-planes. One at infinity when \( t = 0 \).

Computer-assisted exploration. For given \( (x, t) \), chose one of the two possible endpoints. Then construct the candidate density \( \rho(\eta) \) and

1. Numerically follow the orbit \( \rho(\eta) \, d\eta < 0 \) from the origin and see whether it makes it to \( \lambda_0 \) safely. This determines whether the band \( I_0^+ \) can exist.

2. If \( I_0^+ \) exists, numerically construct \( \Re(\tilde{\phi}(\lambda)) \) and see where it is negative. Determine whether the contour \( C \) can be closed around \( [0, iA] \) in such a region.
Comparing the two possible endpoints before breaktime:

And after breaktime:
Breakdown of the ansatz: Failure of inequality in the gaps.

“Dual” ansatz: reverse roles of bands and gaps!
Another example of inequality failure in the gaps. No dual ansatz.
Complete scan of the \((x,t)\)-plane:
Modes of Failure of the Ansatz. Phase Transition.

The ansatz can fail at some \((x,t)\) in several ways:

1. The region admitting a gap contour can “pinch off”.
2. A complex zero of \(\rho(\eta)\) can move onto a band.
3. A band can strike the interval \([0, iA]\).
4. The endpoint functions can fail to be analytic.

Apparently the ansatz can be chosen so that case 1 is the mode of failure.

**Theorem 5** If the genus zero ansatz fails at a point \((x_{\text{crit}}, t_{\text{crit}})\) due to the pinching off of a gap at a point \(\hat{\lambda}\) (not in the shadow of \(I_0^+\)) then for \(|x| - |x_{\text{crit}}| < 0\) and small enough in magnitude, a genus two ansatz suffices to generate a complex phase function.