Singular Asymptotics for Nonlinear Dispersive Waves

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Outline

I. The Semiclassical Linear Schrödinger Equation

II. The Role of Nonlinearity

III. Physical Phenomena

IV. The Cubic Nonlinear Schrödinger Equation

A. Singular Asymptotics in the Defocusing Case
B. Singular Asymptotics in the Focusing Case

V. Conclusion

The Semiclassical Linear Schrödinger Equation

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Goal: With $A(x)$ and $S(x)$ given fixed functions, compute asymptotics for the solution $\psi(x,t)$ in the vicinity of fixed x and t in the semiclassical limit of $\epsilon \downarrow 0$.

Exact Solution via Fourier Transform

For any $\epsilon > 0$, the solution to the initial-value problem is

$$\psi(x,t) = \frac{e^{-i\pi \mathrm{sgn}(t)/4}}{\sqrt{2\pi\epsilon|t|}} \int_{-\infty}^{\infty} A(y) e^{i\theta(y)/\epsilon}\, dy$$

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Asymptotic analysis of $\psi(x,t)$ in the limit $\epsilon \downarrow 0$ with x and t held fixed can be accomplished via the method of stationary phase.

Stationary Phase Analysis

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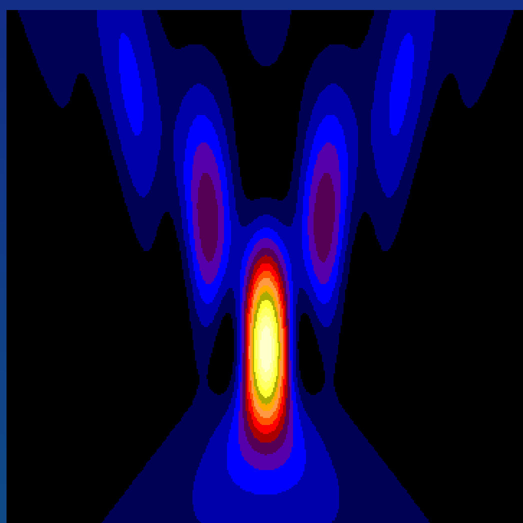
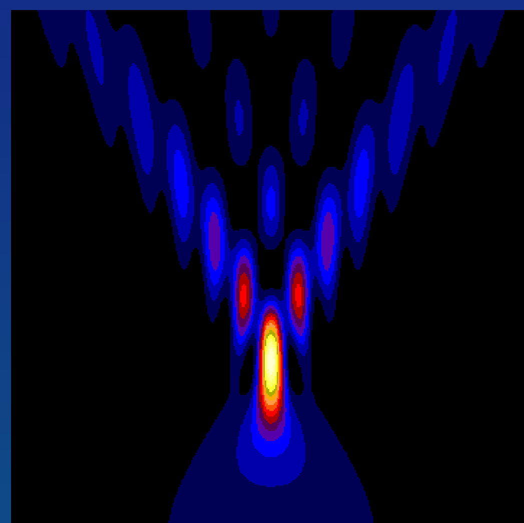
$$\psi(x,t) = \frac{e^{-i\pi\mathrm{sgn}(t)/4}}{\sqrt{|t|}} \sum_{k=1}^{N} \frac{e^{i\pi\mathrm{sgn}(\theta''(y_k))/4}}{\sqrt{|\theta''(y_k)|}} A(y_k) e^{i\theta(y_k)/\epsilon} + O(\epsilon)\,,$$

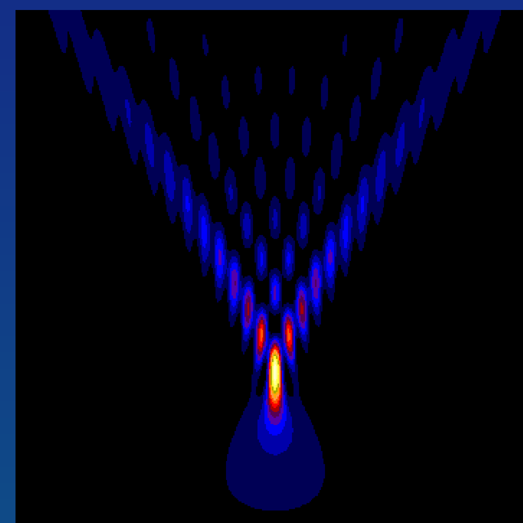
a linear combination of smoothly modulated, rapidly oscillatory exponentials, one for each stationary phase point.

Images of Solutions

Images of $|\psi(x,t)|^2$ over a fixed region of the x $(\leftrightarrow)$ and t $(\updownarrow)$ plane.
Initial conditions: $A(x) = 2\,\mathrm{sech}^2(x)$ and $S(x) = 4\,\mathrm{sech}^2(x)$.


$\epsilon = 0.2$


$\epsilon = 0.1$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Two scales present in the evolution.

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- due to the semiclassical scaling of the PDE.

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Simultaneous presence of microstructure and macrostructure $\Rightarrow$ accurate computation hindered by numerical stiffness.

The Fluid Dynamics Analogy

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Then Schrödinger's equation becomes, exactly,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\epsilon^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial^2 \rho}{\partial x^2} - \left[\frac{1}{2\rho}\frac{\partial \rho}{\partial x}\right]^2\right)$$

with initial data $\rho(x,0) = A(x)^2$ and $u(x,0) = S'(x)$.

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Eikonal equation solved by the method of characteristics. We discover:

> Characteristic lines $x = u_0 t + y_k$ (rays) through a given point $(x, t) = (X, T) \Leftrightarrow$ stationary phase points $y_k(X, T)$.

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- More accurate mathematical modeling: bring in additional terms representing physical phenomena (dominant balance). Especially important when dispersion is weak.
- For conservative dynamics, a natural choice is to include a nonlinear term.

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- Failure of the superposition principle. No Fourier integral representation of the solution of the initial-value problem.
- Bifurcation of the velocity implies phase transitions can occur even if $S'(x) \equiv 0$ (more details later. . .).
- Appearance of solitary waves and solitons.

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Solitary waves: localized traveling waves achieving a dynamical balance between dispersion and nonlinear effects:

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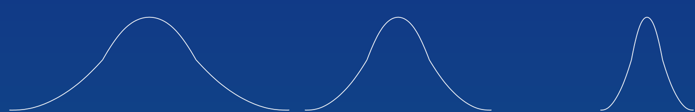

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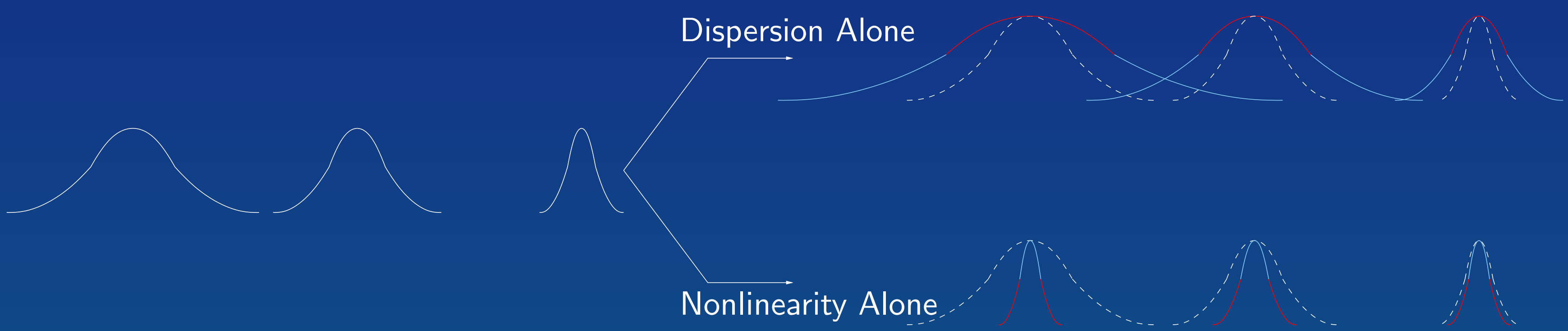

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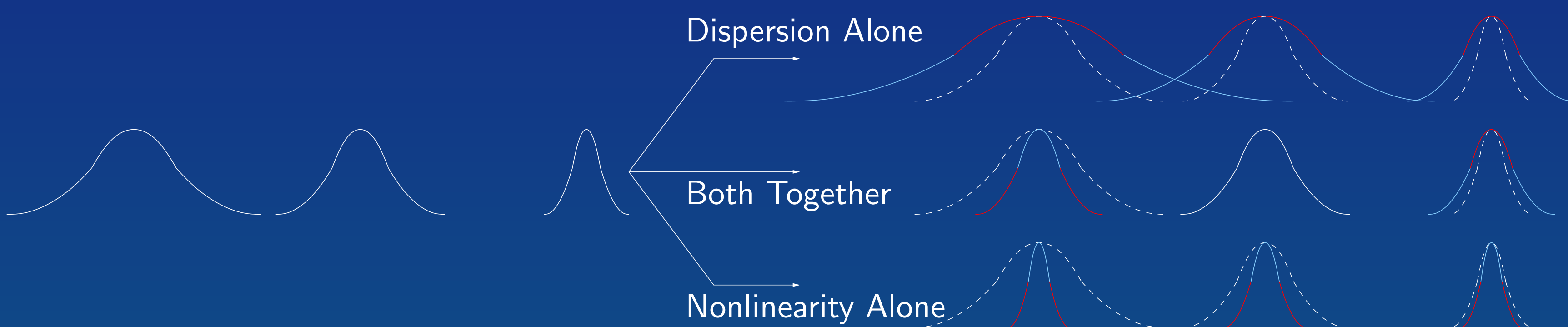

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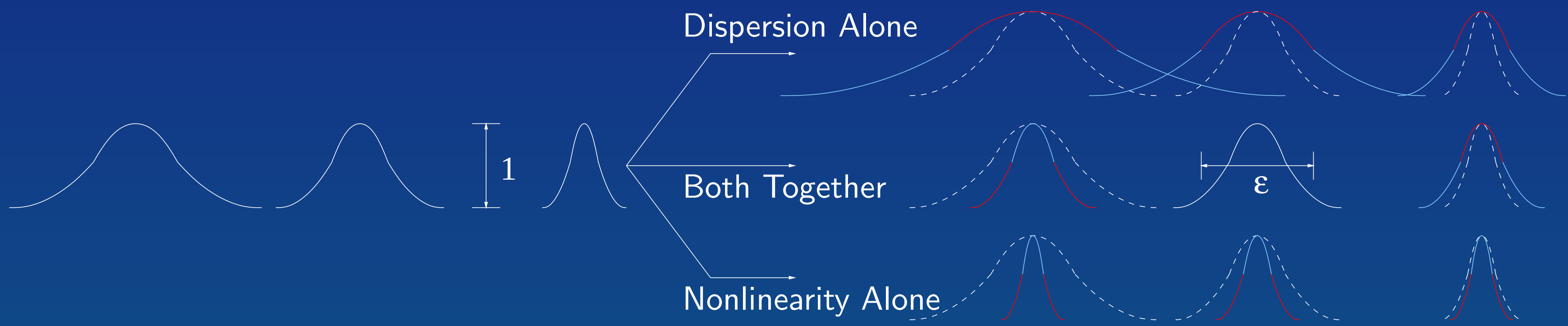

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But how are solitary waves or solitons generated from general initial data? By means of a process that is more dramatic the smaller the dispersion. We are thus led to a study of semiclassical (singular) asymptotics in the nonlinear setting.

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Physical Phenomena

Many interesting physical phenomena involve weak dispersion in a nonlinear setting and the generation of solitary waves and solitons.

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These phenomena may motivate a study of singular asymptotics for nonlinear waves.

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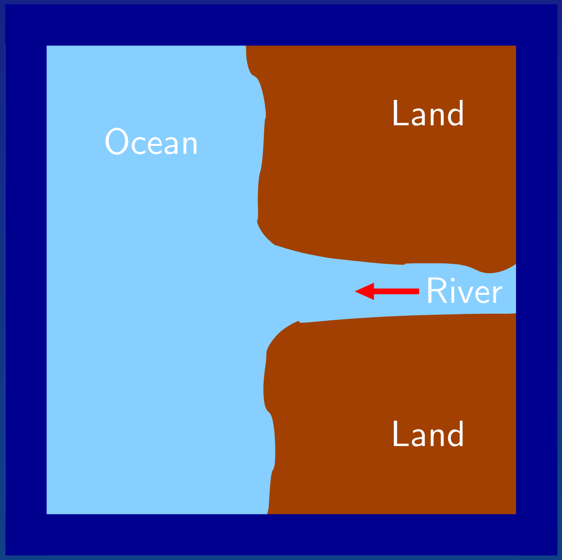

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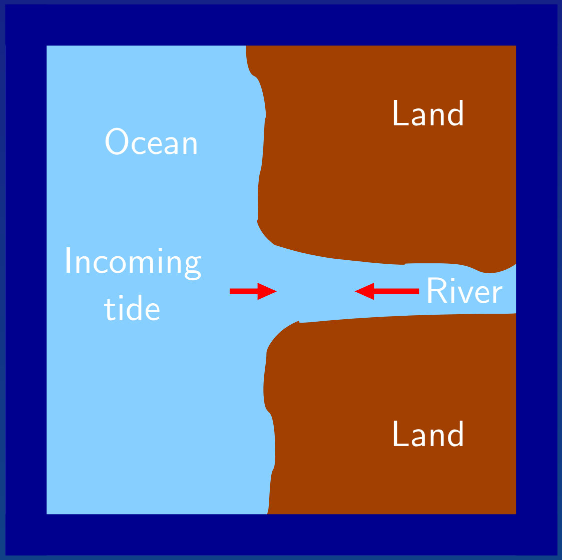

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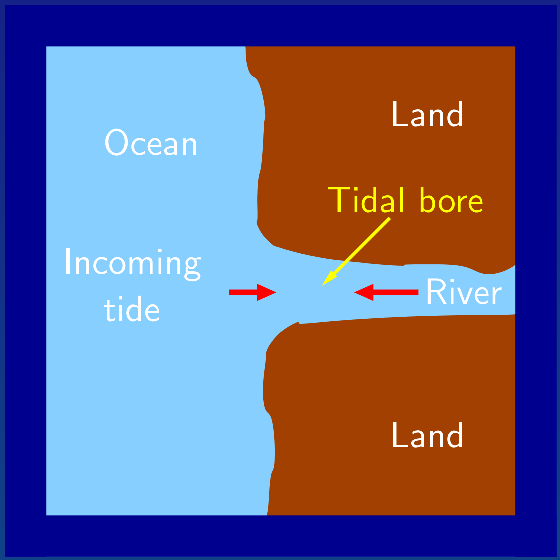

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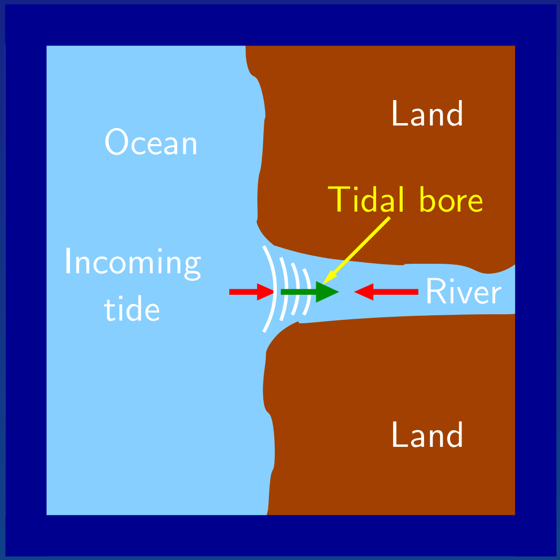

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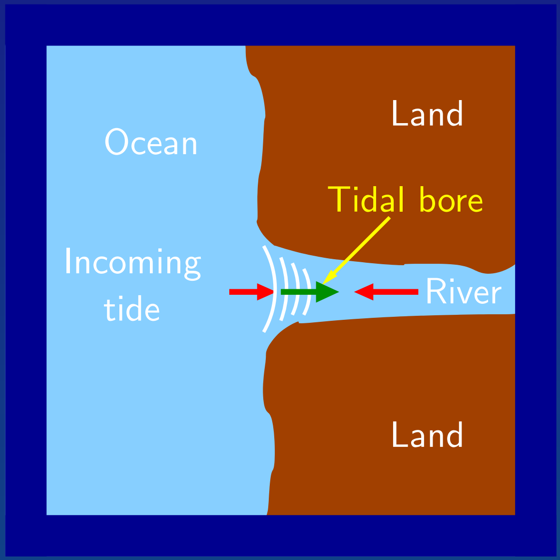

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- Thought to be a solitary wave at the leading edge of an undular bore.

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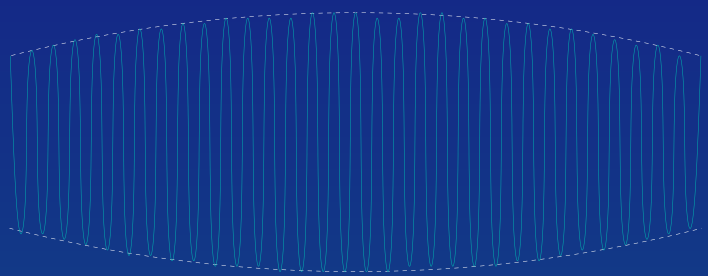
Benjamin-Feir sideband instability of packets of waves in deep water:



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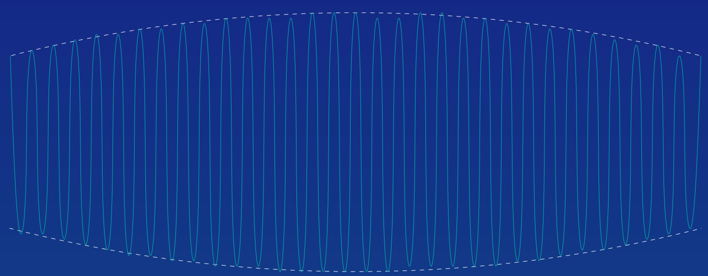
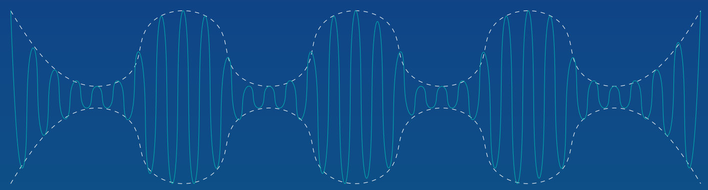


Instability saturates with the formation of structures of size proportional to the dispersion in the system.



Dissipationless Gas Dynamics with Capillarity

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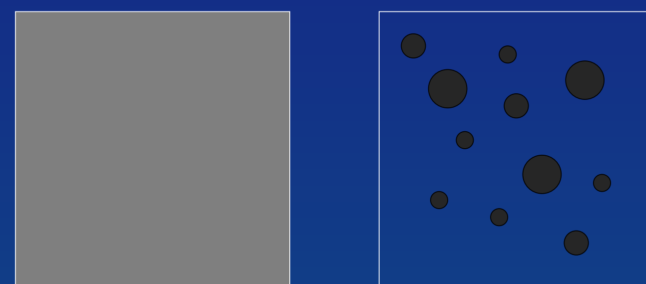

Department of MATHEMATICS University of Michigan

Pulse Propagation in Optical Fibers

"Optical shocks" in fibers with weak normal dispersion:

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The Cubic Nonlinear Schrödinger Equation (NLS)

Two model problems for studying singular asymptotics with nonlinearity:

Defocusing NLS (NLS$-$)

$$i\epsilon\frac{\partial\psi}{\partial t}+\frac{\epsilon^2}{2}\frac{\partial^2\psi}{\partial x^2}-|\psi|^2\psi=0$$

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Focusing NLS (NLS+)

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Consider both posed with initial data of the form: $\psi(x,0)=A(x)e^{iS(x)/\epsilon}$.

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  - ⋆ NLS+ applies to surface wave packets in deep water, capillary gas dynamics with negative pressure (supercooled van der Waals gas), and to pulse propagation in optical fibers with weak anomalous dispersion (among many other phenemena).
- Complete integrability. The existence of a suite of exact solution techniques despite the (strong) nonlinearity.

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Complete Integrability of NLS$\pm$

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$$\epsilon\frac{\partial \mathbf{v}}{\partial x} = \begin{pmatrix} -i\lambda & \psi \\ \mp\psi^* & i\lambda \end{pmatrix}\mathbf{v} \quad \text{(rewrite as } L_\pm^\epsilon[\psi]\mathbf{v} = \lambda\mathbf{v}\text{)}$$

$$i\epsilon\frac{\partial \mathbf{v}}{\partial t} = \begin{pmatrix} \lambda^2 \mp |\psi|^2/2 & i\lambda\psi - \epsilon\psi_x/2 \\ \mp i\lambda\psi^* \mp \epsilon\psi_x^*/2 & -\lambda^2 \pm |\psi|^2/2 \end{pmatrix}\mathbf{v}$$

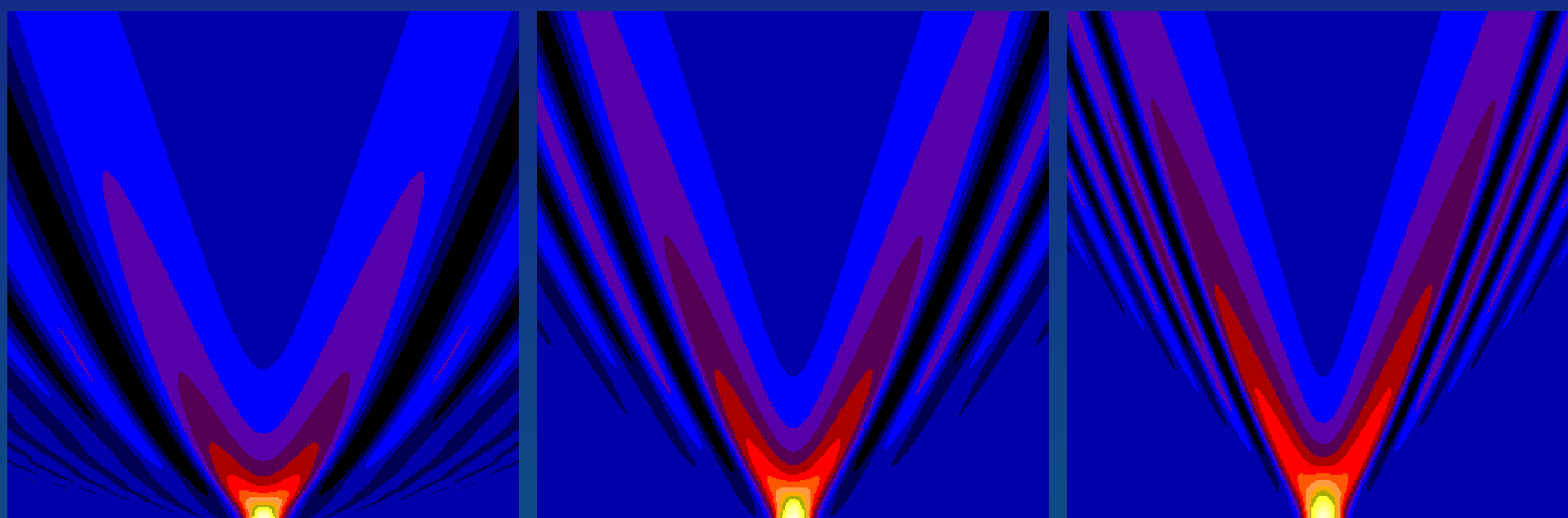
Compatibility demands that $\psi(x,t)$ satisfy NLS$\pm$.

Solution method may be based on spectral theory of the operator $L_\pm^\epsilon[\psi]$.

Back

NLS$-$ in the Semiclassical Limit: Images of Solutions

Images of $|\psi(x,t)|^2$ over a fixed region of the x ($\leftrightarrow$) and t ($\updownarrow$) plane.
Initial conditions: $A(x) = e^{-(x/8)^{10}} + (1-64x^2)_+^2$ and $S(x) \equiv 0$.



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with initial data $\rho(x,0) = A(x)^2$ and $u(x,0) = S'(x)$.

NLS$-$ in the Semiclassical Limit: Formal Expansions

Try $\rho = \rho_0 + \epsilon^2\rho_1 + \ldots$ and $u = u_0 + \epsilon^2 u_1 + \ldots$. Balance powers of ϵ.

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Hyperbolic system with two characteristic velocities: $c_\pm = u_0 \pm \sqrt{\rho_0} \Rightarrow$ singularities may form even if $S'(x) \equiv 0$ (zero initial velocity).

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Valid modeling to set $\epsilon = 0$ and use the fluid equations before breaking?

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Upshot: unlike in the linear theory, "caustics" cannot be deduced from the leading-order problem alone.

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In short, the semiclassical limit for NLS− is a well-posed problem. Back

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Images of $|\psi(x,t)|^2$ over a fixed region of the x ($\leftrightarrow$) and t ($\updownarrow$) plane.
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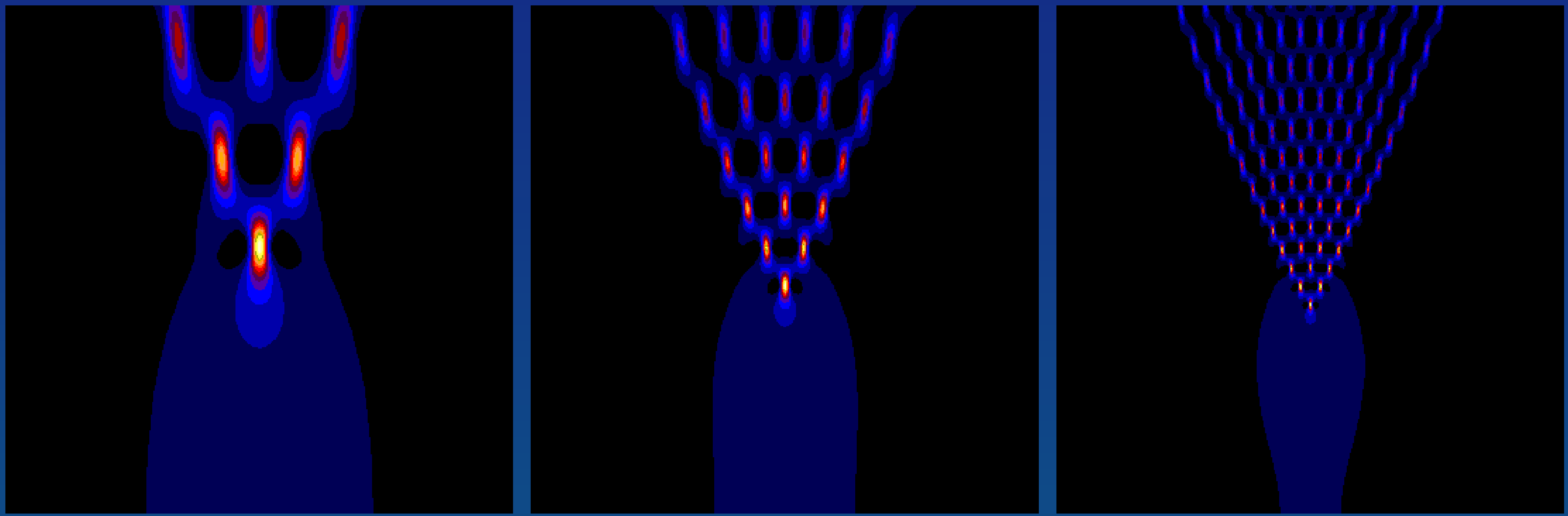

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Elliptic system with complex characteristic velocities: $c_\pm = u_0 \pm i\sqrt{\rho_0} \Rightarrow$ initial-value problem is ill-posed. Solution requires analyticity of $A(x)$ and $S(x)$.

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(Note: eigenvalues $\{\lambda_k\}$ remain fixed!)

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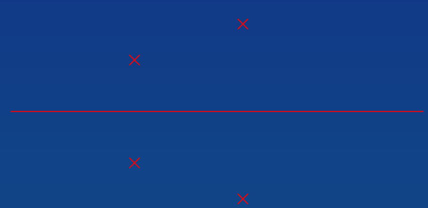

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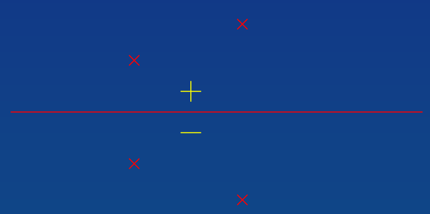

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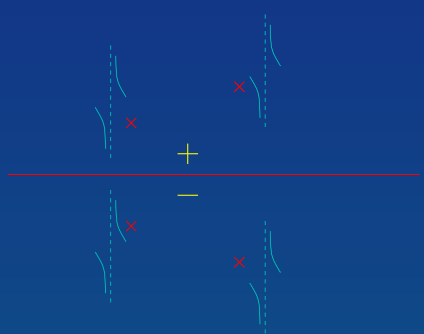

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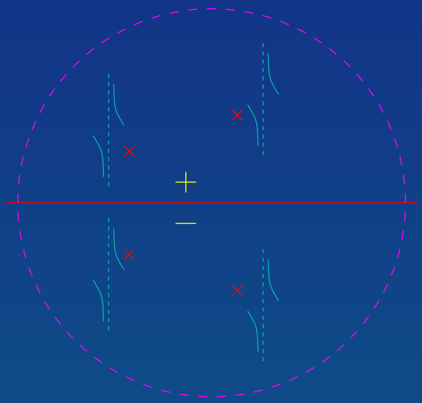

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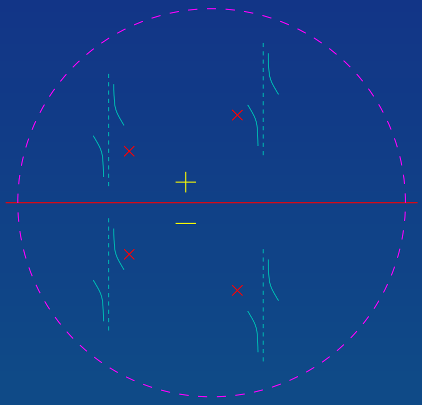

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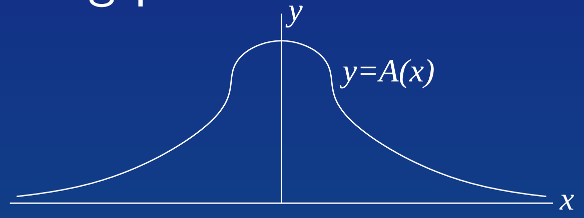
Turning points:



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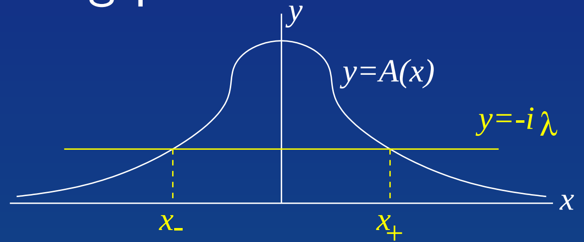

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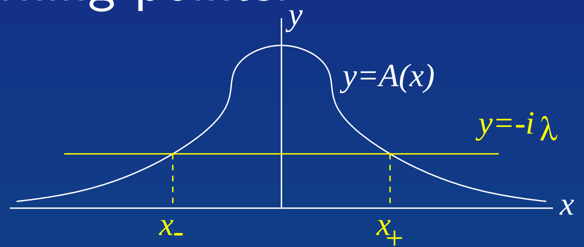

get $x_\pm(\lambda)$ for $\lambda \in [0, iA_{\max}]$.

Scaled density of eigenvalues:

$$\rho^0(\lambda) := \frac{\lambda}{\pi} \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{dx}{\sqrt{A(x)^2 + \lambda^2}}$$

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Correspondence between discrete spectrum and soliton components of the solution motivates the terminology of a semiclassical soliton ensemble for the exact solution of NLS+ associated with this reflectionless approximate data.

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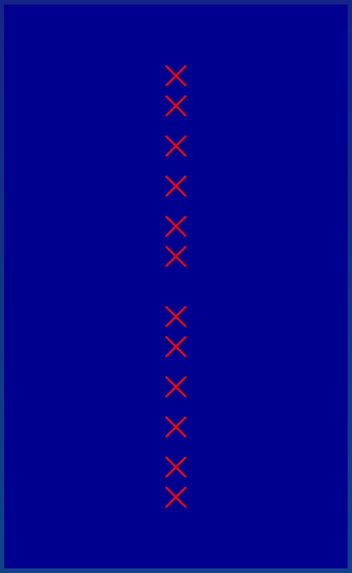
- Strategy: construct $\mathbf{m}(\lambda)$ instead; obtain $\psi(x,t)$ from it after the fact.
- Useful fact: if $A(x)$ is an analytic function, then $\rho^0(\lambda)$ is an analytic function. Thus $\exists$ a natural analytic interpolant for the residues of $\mathbf{m}(\lambda)$ at its poles.

NLS+ in the Semiclassical Limit: Removal of Poles

$\mathbf{m}(\lambda)$ is meromorphic with many poles.



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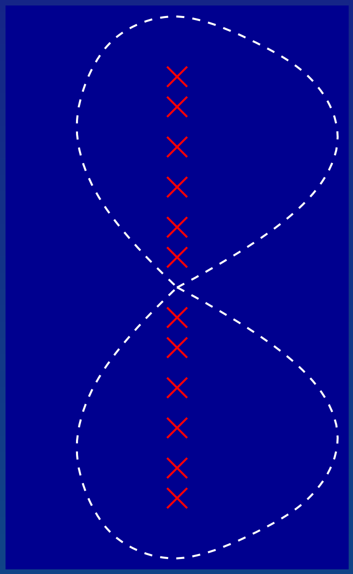
Encircle them with two conjugation-symmetric but otherwise arbitrary contours.



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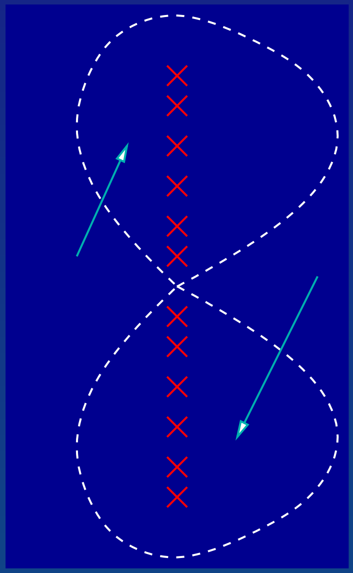

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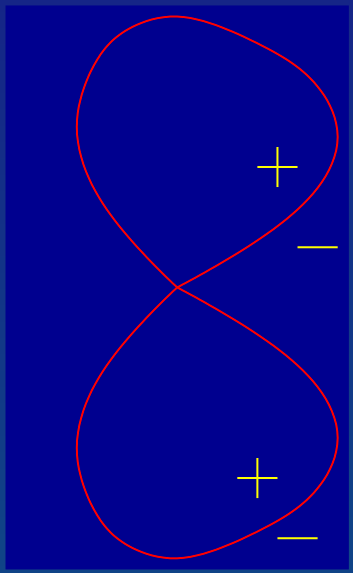

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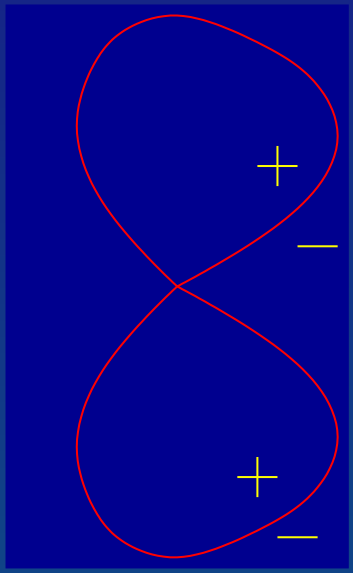

Key idea 1: Advantage of choice of contours is like the steepest-descent or saddle-point method for integration.

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Otherwise, $g(\lambda)$ may be chosen for our convenience.

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- A leading-order approximation of $\mathbf{N}(\lambda)$ may then be built from Riemann Θ-functions.

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$$E[\mu] \quad := \quad \underbrace{\frac{1}{2}\int\int \log\left|\frac{\lambda-\eta^*}{\lambda-\eta}\right| d\mu(\lambda)\, d\mu(\eta)}_{\text{self-energy}} \quad + \quad \underbrace{\int \varphi_{x,t}(\lambda)\, d\mu(\lambda)}_{\text{external field energy}} \, .$$

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This is a generalized Lax-Levermore variational principle for μ.

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Fundamental observation: the semiclassical limit for NLS+ with analytic data is an ill-posed problem.

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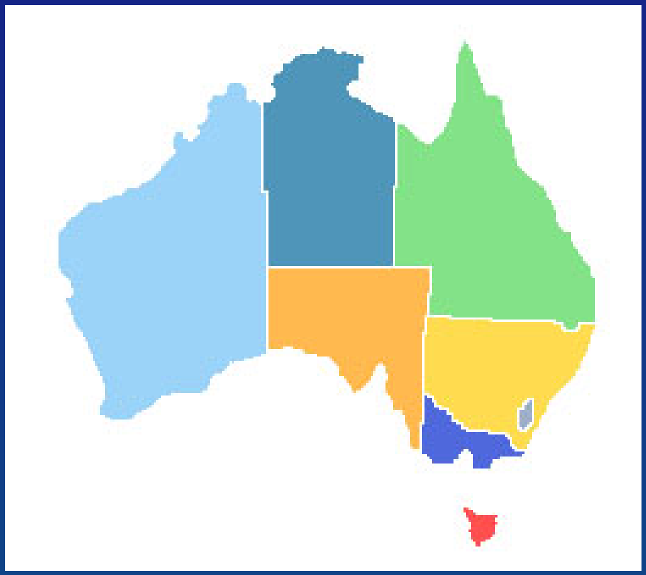
Location of Burketown



Back to Morning Glory