8. Construction of the Lebesgue measure

8.1. Poor man’s version of integration theory. If we try to develop measure theory and integration theory using only the finite (as opposed to countable) additivity property, it is natural to use the following definitions.

Definition 8.1. A collection \( \mathcal{A} \) of subsets of a set \( X \) is said to be an algebra provided:

(i) \( \emptyset \in \mathcal{A} \);
(ii) if \( A \in \mathcal{A} \), then \( X \setminus A \in \mathcal{A} \);
(iii) if \( A, B \in \mathcal{A} \), then \( A \cup B \in \mathcal{A} \).

Remark 8.2. It follows immediately that if \( \mathcal{A} \) is an algebra of subsets of a set \( X \), then \( X \in \mathcal{A} \) and \( \mathcal{A} \) is closed under finite intersections.

Definition 8.3. If \( \mathcal{A} \) is an algebra of subsets of a set \( X \), a map \( \mu : \mathcal{A} \rightarrow [0, +\infty] \) is said to be finitely additive if \( \mu(\emptyset) = 0 \) and given pairwise disjoint sets \( A_1, A_2, \ldots, A_n \in \mathcal{A} \), we have \( \mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j) \).

Now suppose we have a triple \( (X, \mathcal{A}, \mu) \) where \( X \) is a set, \( \mathcal{A} \) is an algebra of subsets of \( X \) and \( \mu : \mathcal{A} \rightarrow [0, +\infty] \) is a finitely additive map. It still makes sense to talk about simple functions in this context, namely, we define a function \( s : X \rightarrow \mathbb{C} \) to be simple if there are sets \( A_1, \ldots, A_n \in \mathcal{A} \) and numbers \( a_1, \ldots, a_n \in \mathbb{C} \) such that \( s = \sum_{j=1}^n a_j \cdot 1_{A_j} \).

We can also define the “integral” of such a function \( s \) with respect to \( \mu \) as \( I_\mu(s) := \sum_{j=1}^n a_j \cdot \mu(A_j) \). Further, we can attempt to define the integral of any “measurable” function \( f : X \rightarrow [0, +\infty] \) as the least upper bound of the set of numbers \( I_\mu(s) \), where \( s \) ranges over all simple functions on \( X \) such that \( 0 \leq s(x) \leq f(x) \) for all \( x \in X \).

The problem with this approach is that since we are not assuming countable additivity, we lose all limit theorems that were so useful for us at the early stages of developing integration theory. In particular, if you look back through the class notes, you will see that the proof of every nontrivial result about integrals relies either explicitly or implicitly on the Monotone Convergence Theorem, which requires the countable additivity of measures in its proof. (In fact, if one assumes MCT, countable additivity follows from it.)

8.2. Carathéodory’s Extension Theorem. Nevertheless, in some situations it is useful to start with a triple \( (X, \mathcal{A}, \mu) \) as above and ask whether we can produce “something countably additive” from this triple. Since measures are defined on \( \sigma \)-algebras, it makes sense to consider \( \sigma(\mathcal{A}) \), the \( \sigma \)-algebra generated by \( \mathcal{A} \). The best possible scenario for us is when we can prove that \( \mu \) can be extended to an honest measure on \( \sigma(\mathcal{A}) \).

Remark 8.4. If \( \mu(X) < \infty \), then, if such an extension of \( \mu \) exists, it must be unique by Dynkin’s Lemma. (To see this, observe that any algebra of sets is, \textit{a fortiori}, a \( \pi \)-system.)

\(^1\)From an intuitive viewpoint, finite additivity is a more natural property than countable additivity.
In general, proving that the desired extension of $\mu$ exists can be a rather nontrivial matter, due to the fact that a general element of $\sigma(\mathcal{A})$ could be a lot more complicated than a general element of $\mathcal{A}$. At least there is an "obvious" obstruction to the existence of an extension. Namely, if we have any chance to extend $\mu$ to a countably additive function on $\sigma(\mathcal{A})$, then $\mu$ must at least satisfy the following property.

**Definition 8.5.** If $\mathcal{A}$ is an algebra of subsets of a set $X$, a map $\mu : \mathcal{A} \rightarrow [0, +\infty]$ is said to be countably additive if $\mu(\emptyset) = 0$ and given a sequence of pairwise disjoint sets $A_1, A_2, A_3, \ldots \in \mathcal{A}$ such that $\bigcup_{n \geq 1} A_n \in \mathcal{A}$, we have $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

The great news is that once this property is satisfied, there is nothing else to worry about. This is the content of the Carathéodory Extension Theorem, whose proof appears on this week’s homework (problems 4.4–4.6). The statement of the theorem is as follows.

**Theorem 8.6** (Carathéodory). Let $X$ be a set, let $\mathcal{A}$ be an algebra of subsets of $X$, let $\Sigma = \sigma(\mathcal{A})$ be the $\sigma$-algebra generated by $\mathcal{A}$, and let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be countably additive. Then there is a measure $\tilde{\mu} : \Sigma \rightarrow [0, +\infty]$ such that $\tilde{\mu}|_{\mathcal{A}} = \mu$.

How do we check countable additivity in practice? The next result is often useful.

**Lemma 8.7.** Let $X$ be a set, $\mathcal{A}$ an algebra of subsets of $X$, and $\mu : \mathcal{A} \rightarrow [0, +\infty]$ a finitely additive map. Assume that $\mu(X) < \infty$. Then $\mu$ is countably additive if and only if, given $\epsilon > 0$ and a sequence $B_1, B_2, B_3, \ldots \in \mathcal{A}$, satisfying $B_1 \supset B_2 \supset \cdots$ and $\mu(B_n) \geq \epsilon$ for all $n \geq 1$, we have $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$.

8.3. **The Lebesgue measure on $(0, 1]$.** Let $X = (0, 1]$ and define $\mathcal{A} \subset 2^X$ to be the collection of all subsets of $X$ the form

\[
S = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_k, b_k],
\]

where $0 \leq a_1 < b_1 < a_2 < \cdots < b_k \leq 1$ and $k \geq 0$ is an integer. Observe that $\mathcal{A}$ is an algebra, and $\mathcal{A}$ generates the $\sigma$-algebra of all Borel subsets of $(0, 1]$.

If $S \in \mathcal{A}$ is of the form (8.1) as above, we put $\mu(S) = \sum_{j=1}^{k} (b_j - a_j)$.

**Proposition 8.8.** The map $\mu : \mathcal{A} \rightarrow [0, +\infty]$ is countably additive.

By Theorem 8.6 and Remark 8.4 there is a unique extension of $\mu$ to a measure on the Borel $\sigma$-algebra of $(0, 1]$. We denote this extension by $m_0$.

8.4. **The Lebesgue measure on $\mathbb{R}$ and on $\mathbb{R}^n$.** If $A \subset \mathbb{R}$ is any Borel subset, we put

\[
m(A) = \sum_{j \in \mathbb{Z}} m_0((A \cap (j, j+1)) - j).
\]

This defines a measure $m$ on the Borel $\sigma$-algebra of $\mathbb{R}$ satisfying $m([a, b]) = b - a$ for all $-\infty < a < b < \infty$. Thus we obtain a construction of the Lebesgue measure on $\mathbb{R}$. Finally, we construct the Lebesgue measure on the Borel $\sigma$-algebra of $\mathbb{R}^n$ by taking the product of $n$ copies of the measure $m$ we just defined.

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2This assumption has to be included because $\mathcal{A}$ may not be closed under countable unions.