Math 632. Homework Set 2

Solutions are due Thursday, February 18th.

As usual, all schemes are assumed of finite type over $k$.

**Problem 1.** Let $X$ be a scheme, and $\mathcal{F}$ a quasicoherent sheaf on $X$.

i) Show that if $(\mathcal{F}_i)_i$ is a family of $\mathcal{O}_X$-submodules of $\mathcal{F}$, then we have an $\mathcal{O}_X$-submodule $\bigcap_i \mathcal{F}_i$ of $\mathcal{F}$ given by

$$\Gamma(U, \bigcap_i \mathcal{F}_i) = \bigcap_i \Gamma(U, \mathcal{F}_i).$$

ii) Show that if all $\mathcal{F}_i$ are quasicoherent, and the family is finite, then $\bigcap_i \mathcal{F}_i$ is quasi-coherent. Give an example to show that this is not necessarily true if the family is infinite.

iii) Use this to give a new proof of the fact that if $(Y_i)_i$ is a finite family of closed subschemes of $X$, then there is a unique smallest closed subscheme $\bigcup_i Y_i$ of $X$ with the property that each $Y_i$ is a closed subscheme of $\bigcup_i Y_i$.

**Problem 2.** Let $X$ be a scheme, and $\mathcal{F}$ a quasicoherent sheaf on $X$.

i) Suppose that $(\mathcal{F}_i)_i$ is a family of quasicoherent $\mathcal{O}_X$-submodules of $\mathcal{F}$. Show that if $\sum_i \mathcal{F}_i$ is the sheaf associated to the presheaf $\mathcal{P}$ given by $\Gamma(U, \mathcal{P}) = \sum_i \Gamma(U, \mathcal{F}_i)$, then $\sum_i \mathcal{F}_i$ is a quasicoherent $\mathcal{O}_X$-submodule of $\mathcal{F}$. Furthermore, show that if $U \subseteq X$ is an affine open subset, then $\Gamma(U, \sum_i \mathcal{F}_i) = \sum_i \Gamma(U, \mathcal{F}_i)$.

ii) Use this to give a new proof of the fact that if $(Y_i)_i$ is a family of closed subschemes of $X$, then there is a unique largest closed subscheme $\bigcap_i Y_i$ of $X$ with the property that it is a closed subscheme of each of the $Y_i$.

**Problem 3.** Let $X$ be a scheme. Show that if

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules on $X$ and if $\mathcal{F}'$ and $\mathcal{F}''$ are (quasi)coherent, then so is $\mathcal{F}$.

**Problem 4.** Prove that if $f : X \rightarrow Y$ is a morphism of schemes, and if $\mathcal{F}$ is a quasicoherent sheaf on $X$, and $\mathcal{E}$ is a locally free sheaf on $Y$, then there is a canonical isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{E})) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

This is the **projection formula**.

**Problem 5.** Let $X$ be a topological space. Show that if

$$0 \rightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \rightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules on $X$ and if $\mathcal{F}'$ and $\mathcal{F}''$ are (quasi)coherent, then so is $\mathcal{F}$.
is an exact sequence of sheaves of abelian groups on $X$ such that $\mathcal{F}'$ is flasque, then the following sequence is exact

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$  

(Hint: given $s'' \in \Gamma(X, \mathcal{F}'')$, consider the set

$$\{(U, s) \mid U \text{ open in } X, s \in \mathcal{F}(U), v(s) = s''|_{U}\}$$

ordered by $(U, s) \leq (V, t)$ if $U \subseteq V$ and $t|_{U} = s$. Show that this set has a maximal element and that if $\mathcal{F}'$ is flasque and $(V, t)$ is maximal, then $V = X$).

**Problem 6.** Let $(X, \mathcal{O}_X)$ be a ringed space.

1. If $U$ is an open subset of $X$ and $i: U \to X$ the inclusion, show that there is a sub-$\mathcal{O}_X$-module $i_!(\mathcal{O}_U)$ of $\mathcal{O}_X$ such that $i_!(\mathcal{O}_U)(V) = \mathcal{O}_X(V)$ if $V \subseteq U$, and $i_!(\mathcal{O}_U)(V) = 0$, otherwise.
2. Show that if $x \in U$, then we have a canonical isomorphism $\mathcal{O}_{U,x} \simeq i_!(\mathcal{O}_U)_x$ and if $x \notin U$, then $i_!(\mathcal{O}_U)_x = 0$. The sheaf $i_!(\mathcal{O}_U)$ is called the *extension by zero* of $\mathcal{O}_U$.
3. Show that for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism

$$\text{Hom}_{\mathcal{O}_X}(i_!(\mathcal{O}_U), \mathcal{F}) \simeq \mathcal{F}(U).$$

4. Deduce that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then it is flasque.
5. Give an example of a scheme $X$ and an open subset $U$ such that $i_!(\mathcal{O}_U)$ is not quasicoherent.