Homework Set 4

Solutions are due Monday, November 23rd.

As usual, all our schemes are assumed to be of finite type over an algebraically closed field \( k \).

**Problem 1.**

i) Let \( f : Y \to X \) be a closed immersion of schemes. Show that for every scheme \( Z \), the induced map \( \text{Hom}(Z,Y) \to \text{Hom}(Z,X) \) (where we denote by \( \text{Hom} \) the set of scheme morphisms) is injective.

ii) Show that if \( f \) is as above, and if \( g : Z \to X \) is a morphism of schemes such that the set-theoretic image of \( g \) is contained in the set-theoretic image of \( f \), then it does not necessarily follow that there is a morphism \( h : Z \to Y \) such that \( f \circ h = g \).

**Problem 2.** Given two closed immersions \( i_1 : Y_1 \to X \) and \( i_2 : Y_2 \to X \), we put \( i_1 \leq i_2 \) if there is a morphism of schemes \( f : Y_1 \to Y_2 \) such that \( i_2 \circ f = i_1 \) (note that in this case \( f \) is unique by the previous problem). This defines an order relation on the closed immersions into \( X \).

i) Show that if \( f \) is as above, then \( f \) is a closed immersion.

ii) Show that if \( i_1 \leq i_2 \) and \( i_2 \leq i_1 \), then the above \( f \) is an isomorphism (in this case, we say the \( i_1 \) and \( i_2 \) are equivalent).

iii) Show that we can identify the closed subschemes of \( X \) with the equivalence classes with respect to this equivalence relation.

**Problem 3.**

i) If \( Y \) is a closed subscheme of \( X \), and if \( U \) is open in \( X \), then \( U \cap Y \) has a natural scheme structure as open subscheme of \( Y \). Show that with this scheme structure, \( U \cap Y \) is a closed subscheme of \( U \).

ii) A morphism \( f : Z \to X \) is a locally closed immersion if it factors as \( Z \xrightarrow{i} W \xrightarrow{j} Y \), where \( i \) is a closed immersion and \( j \) is an open immersion. Deduce from i) that the class of locally closed immersions is closed under composition.

**Problem 4.** Prove the following criterion for gluing closed subschemes. Suppose that \( X \) is a scheme, and that we have an open cover \( X = \bigcup_i U_i \). Suppose that for every \( i \) we have a closed subscheme \( Y_i \) of \( U_i \), such that for all \( i \) and \( j \) we have \( Y_i \cap U_j = Y_j \cap U_i \) (as closed subschemes of \( U_i \cap U_j \)). Show that in this case there is a unique closed subscheme \( Y \) of \( X \) such that for every \( i \), \( Y \cap U_i = Y_i \) as closed subschemes of \( U_i \).
Problem 5. Let $X$ be a scheme, and $Y \subseteq X$ a closed subset. 

i) Show that there is at most one closed subscheme of $X$ that is reduced, and whose support is equal to $Y$.

ii) Show that there is a closed subscheme of $X$, denoted $Y_{\text{red}}$, as in i). Hint: show this first for the affine open subsets of $X$, and then glue the corresponding closed subschemes using the previous problem, and i).

iii) Show that given any closed subscheme $Z$ of $X$ whose support contains $Y$, we have $Y_{\text{red}} \leq Z$ (in the sense of Pb. 2).

iv) If $Y'$ is another closed subscheme of $X$ with support $Y$, then we have a surjection of sheaves $\mathcal{O}_{Y'} \to \mathcal{O}_{Y_{\text{red}}}$. Show that there is an isomorphism of $\mathcal{O}_{Y_{\text{red}}}$ with the image of the morphism of sheaves $\phi: \mathcal{O}_{Y'} \to \mathcal{C}_Y$, where $\mathcal{C}_Y$ is the sheaf of continuous functions on $Y$ with values in $k$, and $\phi(u) = \tilde{u}$.

Problem 6. Show that taking $X$ to $X_{\text{red}}$ extends to a functor from the category of schemes to itself.

Problem 7. Let $Y$ be a scheme and $\{Y_\alpha\}_\alpha$ a family of closed subschemes of $Y$. Show that there is a unique closed subscheme of $Y$ that is contained in all $Y_\alpha$ and which is maximal with this property. This is usually denoted by $\cap_\alpha Y_\alpha$. Hint: do the construction locally, and use the gluing method from Problem 4).

Problem 8. Let $Y_1, \ldots, Y_n$ be closed subschemes of a scheme $Y$. Show that there is a unique minimal element in the set of all closed subschemes of $Y$ that contain all the $Y_i$. This is denoted by $Y_1 \cup \ldots \cup Y_n$. Hint: use the same method as in the previous problem.