Math 632. Homework Set 4

Solutions are due Tuesday, April 6.

All our schemes are of finite type over an algebraically closed field $k$.

**Problem 1.** Show that every automorphism $\phi: \mathbb{P}^n \to \mathbb{P}^n$ is linear, that is, it is induced by an element of $PGL_n$.

**Problem 2.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles on the scheme $X$.

i) Show that if $\mathcal{L}_1$ is ample and $\mathcal{L}_2$ is generated by global sections, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample.

ii) Show that if $\mathcal{L}_1$ is ample, then for any $\mathcal{L}_2$ we have $\mathcal{L}_1^m \otimes \mathcal{L}_2$ ample if $m \gg 0$.

iii) Show that if both $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, then so is $\mathcal{L}_1 \otimes \mathcal{L}_2$.

iv) Show that if $\mathcal{L}_1$ is very ample, and $\mathcal{L}_2$ is generated by global sections, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample.

v) Show that if $\mathcal{L}_1$ is ample, then $\mathcal{L}_1^\otimes m$ is very ample for $m \gg 0$.

**Problem 3.** Let $X = \mathbb{P}^m \times \mathbb{P}^n$ be a product of projective spaces, and let $p: X \to \mathbb{P}^m$ and $q: X \to \mathbb{P}^n$ be the two projections.

i) Show that every line bundle $\mathcal{L}$ on $X$ is isomorphic to $p^*\mathcal{O}_{\mathbb{P}^m}(a) \otimes q^*\mathcal{O}_{\mathbb{P}^n}(b)$ for unique $a, b \in \mathbb{Z}$ (in this case one says that $\mathcal{L}$ has type $(a, b)$).

ii) Show that a line bundle of type $(a, b)$ is ample if and only if $a, b > 0$.

**Problem 4.** Show that if $X$ and $Y$ are nonsingular complete connected curves (that is, schemes of dimension one), then $X$ and $Y$ are birational if and only if they are isomorphic.

**Problem 5.** Prove that a scheme $X$ is affine if and only if the structure sheaf $\mathcal{O}_X$ is ample.

**Problem 6.**

i) Show that if $f: X \to Y$ is a finite morphism of complete varieties, and $\mathcal{L}$ is an ample line bundle on $Y$, then $f^*(\mathcal{L})$ is ample on $X$.

ii) Deduce that the normalization of a projective variety is projective.

iii) For extra credit, prove the converse of the assertion in i): if $f: X \to Y$ is a finite surjective morphism, and $\mathcal{L}$ is a line bundle on $Y$ such that $f^*(\mathcal{L})$ is ample on $X$, then $\mathcal{L}$ is ample.