Math 632. Homework Set 5

Solutions are due Tuesday, April 20.

All our schemes are of finite type over an algebraically closed field k.

Problem 1. Let $f\colon X \to Y$ be a morphism of schemes. Show that if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\mathcal{E}$ is a locally free sheaf on Y, then we have the following *projection formula* for higher direct image sheaves:

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{E})) \simeq R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} \text{ for all } p \geq 0.$$

Problem 2. Let $X \hookrightarrow \mathbf{P}^n$ be a closed subscheme of dimension d. Show that if $X_1, \ldots, X_r$ are the irreducible components of X of dimension d, then

$$\deg(X) = \sum_{i=1}^{r} \ell(\mathcal{O}_{X,X_i}) \cdot \deg(X_i),$$

where each X_i is considered with its reduced scheme structure, and where $\ell(\mathcal{O}_{X,X_i})$ is the length of the zero-dimensional local ring $\mathcal{O}_{X,X_i}$[1].

Problem 3. Prove that for hypersurfaces in $\mathbf{P}^n$ the two notions of degree that we have so far agree (the one in terms of the degree of the defining polynomial, and the one in terms of the Hilbert polynomial).

Problem 4. Let $X \hookrightarrow \mathbf{P}^n$ be a closed subscheme, and H a hypersurface of degree d in $\mathbf{P}^n$, not containing any associated point of X. Show that if $Y = X \cap H$, and if P_X and P_Y are the Hilbert polynomials of X and Y, respectively, then $P_Y(m) = P_X(m) - P_X(m-d)$. Deduce that $\deg(Y) = d \cdot \deg(X)$.

Problem 5. Show that a closed subscheme of $\mathbf{P}^n$ has degree one if and only if it is a linear subspace.

Problem 6. Let C and C' be curves in $\mathbf{P}^2$ (that is, effective Cartier divisors in $\mathbf{P}^2$). Suppose that C and C' have no common irreducible components.

i) Show that C and C' have finitely many intersection points.

[1]This problem requires some familiarity with the notion of length.

ii) If $p \in C \cap C'$, then the *intersection multiplicity* of C and C' at p is defined as

$$i_p(C, C') := \dim_k(\mathcal{O}_{C\cap C',p})^2.$$

Prove the following theorem of Bézout, describing the number of intersection points of C and C', counted with multiplicities:

$$\sum_{p\in C\cap C'} i_p(C, C') = \deg(C) \cdot \deg(C').$$

[2]Note that this is equal with the length of the local ring $\mathcal{O}_{C\cap C',p}$