Problem session 1

Problem 1. Let $(X, \mathcal{O}_X)$ be a ringed space for which there is a finite open cover $X = U_1 \cup \ldots \cup U_r$, such that each U_i is isomorphic as a ringed space to some $\mathrm{Spec}(A_i)$, where A_i is a finitely generated k-algebra (k is an algebraically closed field). Show that there is a scheme $(Y, \mathcal{O}_Y)$ (in our sense) such that we have an isomorphism of ringed spaces $(X, \mathcal{O}_X) \simeq (t(Y), \mathcal{O}_{t(Y)})$.

Problem 2. Let $M^r_{m,n}(k)$ be the set of all matrices $A \in M_{m,n}(k) = k^{mn}$ of rank $\leq r$.

i) Show that $M^r_{m,n}(k)$ is the affine cone over an irreducible projective variety $Z^r_{m,n}$ in $\mathbf{P}^{mn-1}$.

ii) Show that if we identify $M_{m,n}(k)$ with $\mathrm{Hom}_k(k^m, k^n)$, and if $G(m-r, k^m)$ is the Grassmann variety of $(m-r)$-planes in k^m, then the set

$$\Delta := \{(A, W) \in Z^r_{m,n} \times G(m-r, k^m) \mid W \subseteq \mathrm{Ker}(A)\}$$

is a closed subset of $Z^r_{m,n} \times G(m-r, k^m)$.

iii) Use the two morphisms $p\colon \Delta \to Z^r_{m,n}$ and $q\colon \Delta \to G(m-r, k^m)$ induced by the projections onto the two factors to prove that

$$\mathrm{codim}(Z^r_{m,n}, \mathbb{P}^{mn-1}) = (m-r+1)(n-r+1).$$

Problem 3. Let X and Y be two disjoint closed subsets of $\mathbb{P}^n$. The *join* of X and Y is the union $J(X, Y)$ of all lines $\overline{pq}$ in $\mathbb{P}^n$, where $p \in X$ and $q \in Y$.

i) Show that $J(X, Y)$ is a closed subset of $\mathbb{P}^n$.

ii) Show that $\dim(J(X, Y)) = \dim(X) + \dim(Y) + 1$.