Problem session 1

Problem 1. Let \((X, \mathcal{O}_X)\) be a ringed space for which there is a finite open cover \(X = U_1 \cup \ldots \cup U_r\), such that each \(U_i\) is isomorphic as a ringed space to some \(\text{Spec}(A_i)\), where \(A_i\) is a finitely generated \(k\)-algebra (\(k\) is an algebraically closed field). Show that there is a scheme \((Y, \mathcal{O}_Y)\) (in our sense) such that we have an isomorphism of ringed spaces \((X, \mathcal{O}_X) \cong (t(Y), \mathcal{O}_{t(Y)})\).

Problem 2. Let \(M_{r,m,n}^r(k)\) be the set of all matrices \(A \in M_{m,n}(k) = k^{mn}\) of rank \(\leq r\).

i) Show that \(M_{m,n}^r(k)\) is the affine cone over an irreducible projective variety \(Z_{m,n}^r\) in \(\mathbb{P}^{mn-1}\).

ii) Show that if we identify \(M_{m,n}(k)\) with \(\text{Hom}_k(k^m, k^n)\), and if \(G(m - r, k^m)\) is the Grassmann variety of \((m - r)\)-planes in \(k^m\), then the set

\[
\Delta := \{(A, W) \in Z_{m,n}^r \times G(m - r, k^m) \mid W \subseteq \ker(A)\}
\]

is a closed subset of \(Z_{m,n}^r \times G(m - r, k^m)\).

iii) Use the two morphisms \(p: \Delta \to Z_{m,n}^r\) and \(q: \Delta \to G(m - r, k^m)\) induced by the projections onto the two factors to prove that

\[
\text{codim}(Z_{m,n}^r, \mathbb{P}^{mn-1}) = (m - r + 1)(n - r + 1).
\]

Problem 3. Let \(X\) and \(Y\) be two disjoint closed subsets of \(\mathbb{P}^n\). The join of \(X\) and \(Y\) is the union \(J(X, Y)\) of all lines \(pq\) in \(\mathbb{P}^n\), where \(p \in X\) and \(q \in Y\).

i) Show that \(J(X, Y)\) is a closed subset of \(\mathbb{P}^n\).

ii) Show that \(\dim(J(X, Y)) = \dim(X) + \dim(Y) + 1\).