Problem session 2

Problem 1. Show that if $X$ and $Y$ are topological spaces, with $X$ irreducible, and $f: X \to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible. Use this to show that the closed subset

$$M_{m,n}^r(k) = \{ A \in M_{m,n}(k) \mid \text{rank}(A) \leq r \}$$

of $A_{mn}$ is irreducible.

Problem 2. Let $X$ be a topological space.

i) Show that if $Y$ is a subset of $X$ (with the induced topology), then $Y$ is irreducible if and only if its closure $\overline{Y}$ is irreducible.

ii) Suppose that $X$ is Noetherian, and that $Y$ is a subset $X$. Show that if $Y = Y_1 \cup \ldots \cup Y_r$ is the irreducible decomposition of $Y$, then $\overline{Y} = \overline{Y_1} \cup \ldots \cup \overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

Problem 3.

i) Show that $A^1 \setminus \{0\}$ is an affine variety (recall: this means that it is isomorphic to a closed subset of an affine space).

ii) Let $U = A^2 \setminus \{(0,0)\}$. What is $\mathcal{O}(U)$?

iii) Deduce that $U$ is not an affine variety.

Problem 4. Show that the set

$$B = \left\{ (a, b, c, d) \in A^4 \mid \text{rank} \begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix} \leq 1 \right\}$$

is an irreducible closed subset of $A^4$. Determine its ideal.