Problem 1. A linear algebraic group is an affine variety $G$ that is also a group, and such that the multiplication $\mu: G \times G \to G$, $\mu(g, h) = gh$, and the inverse map $\iota: G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. A morphism of linear algebraic groups is a morphism of algebraic varieties, that is also a group homomorphism.

i) Show that $(k, +)$ and $(k^*, \cdot)$ are linear algebraic groups.

ii) Show that the set $\text{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in $k$ is a linear algebraic group.

iii) Show that if $G$ and $H$ are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) torus $(k^*)^n$ is a linear algebraic group with respect to component-wise multiplication.

Problem 2. Let $G$ be a linear algebraic group acting algebraically on an affine variety $X$ (that is, the map $G \times X \to X$ given by the action is a morphism of algebraic varieties). Show that in this case $G$ has an induced linear action on $\mathcal{O}(X)$. While $\mathcal{O}(X)$ has infinite dimension over $k$, show that the action of $G$ on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace $V$ of $\mathcal{O}(X)$ that is preserved by the $G$-action (Hint: consider the corresponding $k$-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

Problem 3. Let $G$ and $X$ be as in the previous problem. Consider a system of $k$-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \text{GL}_N(k)$, and an isomorphism of $X$ with a closed subset of $\mathbb{A}^N$, such that the action of $G$ on $X$ is induced by the standard action of $\text{GL}_N(k)$ on $\mathbb{A}^N$. Use the same argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\text{GL}_N(k)$.

Problem 4. Show that the linear algebraic group $\text{GL}_m(k) \times \text{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbb{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action? Show that the orbits are locally closed subsets of $M_{m,n}(k)$. N.B. This is a general fact about algebraic group actions.