Problem session 8

While we will not mention this explicitly in what follows, all schemes are assumed to be of finite type over an algebraically closed field \( k \).

**Problem 1.** Let \( X \) and \( Y \) be two locally ringed spaces over \( k \), and let \( X = U_1 \cup \ldots \cup U_r \) be an open cover of \( X \).

i) Show that if \( f, g: X \to Y \) are two morphisms such that \( f|_{U_i} = g|_{U_i} \) for all \( i \), then \( f = g \) (for any open subset \( U \) of \( X \), and any morphism \( h: X \to Y \), we denote by \( h|_U \) the composition of \( h \) with the morphism \( i: U \to X \) induced by inclusion).

ii) Show that if we have morphisms \( h_i: U_i \to Y \) such that \( h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j} \) for all \( i \) and \( j \), then there is a unique morphism \( h: X \to Y \) such that \( h|_{U_i} = h_i \) for every \( i \).

**Problem 2.**

i) Let \( X \) be a scheme, and \( i: W \to X \) an open immersion. Show that if \( f: Y \to X \) is a morphism of schemes such that \( f(Y) \subseteq i(W) \), then there is a unique morphism of schemes \( g: Y \to W \) such that \( i \circ g = f \).

ii) Deduce that if \( f: Y \to X \) is a morphism of schemes, and if \( U \) is an open subscheme of \( X \), then we have an induced morphism of schemes \( f^{-1}(U) \to U \).

**Problem 3.** Let \( f: X \to Y \) be a morphism of schemes. If there is an open cover \( Y = V_1 \cup \ldots \cup V_r \) such that the induced morphism \( f^{-1}(V_i) \to V_i \) is an isomorphism for every \( i \), then \( f \) is an isomorphism.

**Problem 4.** Show that if \( X \) is an affine scheme, then for every scheme \( Y \) the canonical map

\[
\text{Hom}(Y, X) \to \text{Hom}_{k-\text{alg}}(\mathcal{O}(X), \mathcal{O}(Y))
\]

is a bijection. (You may assume you know this when also \( Y \) is affine, as we will prove this in class).