Problem session 8

As usual, all schemes are assumed to be of finite type over an algebraically closed field \( k \).

**Problem 1.** Blow-ups appear naturally when resolving indeterminacies of rational maps, as follows. Suppose that \( \mathcal{L} \) is a line bundle on an integral scheme \( X \), and \( V \subseteq \Gamma(X, \mathcal{L}) \) is a finite dimensional linear subspace, defining the rational map \( \varphi = \varphi_V : X \dashrightarrow \mathbb{P}(V) \).

i) Show that if \( Z \) is the base locus of \( V \) (with the corresponding scheme structure), and if \( \pi : \text{Bl}_Z X \to X \) is the blow-up of \( X \) along \( Z \), then the rational map \( \varphi \circ \pi^{-1} \) is in fact a morphism.

ii) In general, if \( h : X \dashrightarrow Y \) is a rational map between the integral schemes \( X \) and \( Y \), with \( Y \) separated, then the graph \( \Gamma_h \) of \( h \) is defined as follows: if \( U \subset X \) is an open subset of \( X \) on which \( h \) is defined, then \( \Gamma_h \) is the closure in \( X \times Y \) of the graph of \( h : U \to Y \) (check that this definition is independent of \( U \)). Show that if \( \varphi \) is as above, then \( \text{Bl}_Z(X) \) is isomorphic to the graph of \( \varphi \).

**Problem 2.** If \( Y \) is a scheme, and \( y \in Y \) is a point defined by \( m_y \), then the (abstract) tangent cone of \( Y \) at \( y \) is \( C_y Y := \text{Spec}(\oplus_{i \geq 0} m^i/m^{i+1}) \), and the projectivized tangent cone of \( Y \) at \( y \) is \( P(C_y Y) := \text{Proj}(\oplus_{i \geq 0} m^i/m^{i+1}) \) (hence \( P(C_y Y) \) is isomorphic to the inverse image of \( y \) in \( \text{Bl}_y(Y) \)). Suppose now that \( Y \) is a closed subscheme of \( X = \mathbb{A}^n \) (more generally, a similar discussion holds if we only assume \( X \) nonsingular).

i) Show that we have a closed immersion \( P(C_y Y) \hookrightarrow P(C_y \mathbb{A}^n) \cong \mathbb{P}^{n-1} \). The affine cone over the image is the embedded tangent cone to \( Y \) at \( y \).

ii) Show that the tangent cone of \( Y \) at \( y \) has dimension equal to \( \dim(\mathcal{O}_{Y,y}) \).

iii) Show that if \( Y \) is a hypersurface in \( \mathbb{A}^n \) defined by \( f = 0 \), and if \( f = f_m + f_{m+1} + \ldots + f_d \), with \( \deg(f_i) = i \) and \( f_m \neq 0 \), then \( C_0 Y \) is defined in \( \mathbb{A}^n \) by the ideal \( (f_m) \).

iv) Show that more generally, if for \( f \) as above we put \( \text{in}(f) = f_m \), then for every closed subscheme \( Y \) of \( \mathbb{A}^n \) the ideal defining \( C_0 Y \) is generated by those \( \text{in}(f) \) for all \( f \) in the ideal defining \( Y \) (note: it is not enough to only consider a system of generators of the ideal defining \( Y \)).

v) Show that the embedded tangent cone of \( Y \) at \( y \) is contained in the tangent space of \( Y \) at \( y \).