Problem session 9

While we will not mention this explicitly in what follows, all schemes are assumed to be of finite type over an algebraically closed field $k$.

**Problem 1.** Show that if $(X, \mathcal{O})$ is a reduced scheme and if $U \subseteq X$ is a dense open subset, then the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective.

**Problem 2.** Let $(X, \mathcal{O})$ be a scheme and $f \in \mathcal{O}(X)$. Recall that we have a corresponding continuous function $\tilde{f} : X \to k$, such that $\tilde{f}(x)$ is the class of $f_x \in \mathcal{O}_x$ in the quotient modulo the maximal ideal, which is canonically isomorphic to $k$. Let

$$X_f := \{ x \in X \mid \tilde{f}(x) \neq 0 \}.$$

i) Describe this set when $X = \text{Specm}(R)$.

ii) Show that the restriction map induces a morphism of $k$-algebras $\mathcal{O}(X)_f \to \mathcal{O}(X_f)$.

Prove that for every $X$, this is an isomorphism.

**Problem 3.** For an arbitrary scheme $X$, use the previous problem and the canonical morphism

$$X \to \text{Spec} \mathcal{O}(X)$$

to prove the following criterion for $X$ to be affine: if $f_1, \ldots, f_r \in \mathcal{O}(X)$ are such that they generate the unit ideal in $\mathcal{O}(X)$ and all $X_{f_i}$ are affine schemes, then $X$ is affine.

**Problem 4.**

i) Show that if $X_1, \ldots, X_n$ are schemes, then on the disjoint union $\bigsqcup_{i=1}^n X_i$ there is a unique scheme structure (up to a canonical isomorphism) such that each inclusion $X_i \subseteq X$ gives an open immersion.

ii) Show that for every scheme $X$, its connected components are open in $X$.

iii) Show that $\text{Specm}(R)$ is disconnected if and only if there is an isomorphism $R \simeq R_1 \times R_2$ for suitable $k$-algebras $R_1$ and $R_2$. 

1