We consider the question: “How bad can the deformation space of an object be?” (Alternatively: “What singularities can appear on a moduli space?”) The answer seems to be: “Unless there is some a priori reason otherwise, the deformation space can be arbitrarily bad.” We show this for a number of important moduli spaces.

More precisely, up to smooth parameters, every singularity that can be described by equations with integer coefficients appears on moduli spaces parameterizing: smooth projective surfaces (or higher-dimensional manifolds); smooth curves in projective space (the space of stable maps, or the Hilbert scheme); plane curves with nodes and cusps; stable sheaves; isolated threefold singularities; and more. The objects themselves are not pathological, and are in fact as nice as can be. This justifies Mumford's philosophy that even moduli spaces of well-behaved objects should be arbitrarily bad unless there is an a priori reason otherwise.

I will begin by telling you what “moduli spaces” and “deformation spaces” are. The complex-minded listener can work in the holomorphic category; the arithmetic listener can think in mixed or positive characteristic. This talk is intended to be (mostly) comprehensible to a broad audience.