**Homework Set 10**

Solutions are due Friday, December 7th.

**Problem 1.** Let \( f : X \to Y \) be a dominant morphism between irreducible algebraic varieties. One says that \( f \) is *generically finite* if there are nonempty open subsets \( U \subseteq X \) and \( V \subseteq Y \) such that \( f \) induces a finite morphism \( U \to V \).

1) Show that \( f \) is generically finite if and only if \( \dim(X) = \dim(Y) \).
2) Show that if \( f \) is generically finite, then in fact there is a nonempty open subset \( V \subseteq Y \) such that the induced morphism \( f^{-1}(V) \to V \) is finite.

**Problem 2.** Let \( X \) and \( Y \) be algebraic varieties, and \( x \) and \( y \) be points on \( X \) and \( Y \), respectively.

1) Show that there is a canonical isomorphism \( T_{x,y}X \times Y \cong T_xX \times T_yY \).
2) Deduce that \( (x, y) \in X \times Y \) is a nonsingular point if and only if \( x \in X \) and \( y \in Y \) are both nonsingular points.

**Problem 3.** Let \( G \) be a linear algebraic group acting on the variety \( X \). Show that every orbit of \( G \) in \( X \) is nonsingular.

The following is a very useful interpretation of the tangent space at a point.

**Problem 4.** Let \( X \) be an affine algebraic variety, and \( x \in X \) a point. Show that the tangent space \( T_xX \) is in natural bijection with the set of \( k \)-algebra homomorphisms \( f : \mathcal{O}(X) \to k[t]/(t^2) \) with the property that if \( p : k[t]/(t^2) \to k \) is the canonical surjection, then \( p \circ f \) is the map to \( k \) corresponding to \( x \in X \).

**Problem 5.** Recall that \( D_r(m, n) \subseteq M_{m,n}(k) \) denotes the set of matrices \( A \) such that \( \text{rk}(A) \leq r \).

1) Show that the group \( \text{Gl}_m(k) \times \text{Gl}_n(k) \) has a natural action on \( M_{m,n}(k) \) such that the orbits are the sets \( D_r(m, n) \setminus D_{r-1}(m, n) \). Deduce that every point in \( D_r(m, n) \setminus D_{r-1}(m, n) \) is a nonsingular point of \( D_r(m, n) \).
2) Let \( A = (a_{ij}) \in D_r(m, n) \). Show that \( T_A D_r(m, n) \) is isomorphic to the vector space of matrices \( A + tB \in M_{m,n}(k[t]/(t^2)) \), having all \( (r + 1) \)-minors equal to zero.
3) Deduce that if \( A \in D_{r-1}(m, n) \), then \( \dim_k T_A D_r(m, n) = mn \), hence \( A \) is a singular point of \( D_r(m, n) \).