Homework Set 6

Solutions are due Wednesday, October 31st.

Problem 1. Let $X$ and $Y$ be algebraic prevarieties, and let $P$ and $Q$ be points on $X$ and $Y$, respectively.

i) Show that if $\phi: \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ is a local ring homomorphism, then there are open subsets $U \subseteq X$ and $V \subseteq Y$, and a morphism $f: U \to V$ with $f(P) = Q$, such that $f$ induces $\phi$.

2) Deduce that if $\phi$ as above is an isomorphism, then after possibly replacing $U, V$ by $U' \subseteq U$ and $V' \subseteq V$, we have that $f$ is an isomorphism.

Problem 2. Let $f: X \dashrightarrow Y$ be a rational map between the irreducible varieties $X$ and $Y$. The graph $\Gamma_f$ of $f$ is defined as follows. If $U$ is an open subset of $X$ such that $f$ is defined on $U$, then the graph of $f|_U$ is well-defined, and it is a closed subset of $U \times Y$. By definition, $\Gamma_f$ is the closure of the graph of $f|_U$ in $X \times Y$.

i) Show that the definition is independent of the choice of $U$.

ii) Let $p: \Gamma_f \to X$ and $q: \Gamma_f \to Y$ be the morphisms induced by the two projections. Show that $p$ is a birational morphism, and that $q$ is birational if and only if $f$ is.

iii) Show that if the fiber $p^{-1}(x)$ does not consist of only one point, then $f$ is not defined at $x \in X$.

Problem 3. A plane Cremona transformation is a birational map of $\mathbb{P}^2$ into itself. Consider the following example of quadratic Cremona transformation: $\phi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $\phi(x: y: z) = (yz: xz: xy)$, when no two of $x, y, z$ are zero.

1) Show that $\phi$ is birational, and its own inverse.

2) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\phi$ induces an isomorphism $U \simeq V$.

3) Describe the open sets on which $\phi$ and $\phi^{-1}$ are defined.

Problem 4. Generalize the construction of the blowing-up to dimension $\geq 2$, as follows. Thinking of $\mathbb{P}^{n-1}$ as the set of lines in $\mathbb{A}^n$, define the blowing-up of $\mathbb{A}^n$ at $0$ as the set

$$\text{Bl}_0(\mathbb{A}^n) := \{(P, \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} | P \in \ell\}.$$ 

1) Show that $\text{Bl}_0(\mathbb{A}^n)$ is a closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

2) Show that the restriction of the projection onto the first component gives a morphism $\pi: \text{Bl}_0(\mathbb{A}^n) \to \mathbb{A}^n$ that is an isomorphism over $\mathbb{A}^n \smallsetminus \{0\}$.

3) Show that $\pi^{-1}(0) \simeq \mathbb{P}^{n-1}$.

4) Show that $\pi$ is a closed map.

5) Show that $\text{Bl}_0(\mathbb{A}^n)$ can be described as the graph of a suitable rational map.