Homework Set 1

Please, try to do all of the following problems. Solutions to two of them are due on Friday January 20.

Problem 1. The goal of this problem is to describe the open subschemes of a scheme.

i) Show that if \((X, \mathcal{O}_X)\) is a locally ringed space, then for every open subset \(U \subseteq X\) we have a canonical locally ringed space \((U, \mathcal{O}_U)\) supported on \(U\).

ii) Show that if \(X = \text{Spec } A\) and if \(U = D(f) = \{p \in \text{Spec } A \mid f \notin p\}\) for some \(f\) in \(A\), then \(U\) with the above ringed space structure is isomorphic to \(\text{Spec } A_f\).

iii) Deduce that if \(X\) is a scheme and \(U \subseteq X\) is open, then \((U, \mathcal{O}_U)\) is a scheme such that the natural the set-theoretic inclusion \(U \rightarrow X\) can be extended to a morphism of schemes \(i_U\).

iv) Show that if \(U\) is as above and if \(f: Y \rightarrow X\) is a morphism of schemes, then \(f\) factors as \(i_U \circ g\) for some morphism of schemes \(g: Y \rightarrow U\) if and only if set-theoretically we have \(\text{Im}(f) \subseteq U\) (in this case \(g\) is uniquely determined).

Remark. A morphism of schemes \(f: Y \rightarrow X\) is an open immersion if it factors as \(i_u \circ g\), for some \(U\) as above and an isomorphism of schemes \(g: Y \rightarrow U\).

Problem 2. Let \(f: Y \rightarrow X\) be a morphism of schemes. Show that if there is an open cover \(X = \bigcup_i U_i\) such that the induced morphisms \(f^{-1}(U_i) \rightarrow U_i\) are isomorphisms, then \(f\) is an isomorphism (this says that the notion of isomorphism is local on the base).

Problem 3. Recall that a topological space \(Y\) is irreducible if whenever \(Y = A \cup B\), with \(A\) and \(B\) closed, we have \(A = Y\) or \(B = Y\).

i) Show that for every topological space \(Y\), if \(y\) is a point in \(Y\), then the set \(\{y\}\) is irreducible (with the induced topology).

ii) Show that if \(X\) is a scheme and \(Z \subseteq X\) is an irreducible closed subset, then \(Z\) has a unique generic point, i.e. a point \(x\) such that \(Z = \overline{\{x\}}\).

iii) Show that if \(f: X \rightarrow Y\) is a morphism of schemes and if \(x\) is the generic point of the closed subset \(Z \subseteq X\), then \(f(x)\) is the generic point of \(f(Z)\).

Remark. Generic points provide a convenient tool. The general idea is that the behavior at the generic point of \(Z\) translates in the behavior on some open subset of \(Z\). Existence of generic points is one of the advantages of working with schemes.

Problem 4. Describe the closed points of \(X = \text{Spec } \mathbb{R}[x_1, \ldots, x_n]\). What are the \(\mathbb{C}\)-valued and the \(\mathbb{R}\)-valued points of \(X\)?
Problem 5. Let $f : X \to Y$ be a morphism of schemes, $y$ a point in $Y$ and $i : X_y \to X$ the fiber over $y$, i.e. we have a fiber product diagram

$$
\begin{array}{ccc}
X_y & \xrightarrow{j} & X \\
\downarrow & & \downarrow f \\
\text{Spec } k(y) & \to & Y
\end{array}
$$

Show that at the level of topological spaces, $j$ gives a homeomorphism onto its image, the set $\{ x \in X \mid f(x) = y \}$. 