Homework Set 10

Please, try to do all of the following problems. Solutions to three of them are due on Monday April 10.

**Problem 1.** Let $X$ be a Noetherian scheme.

1) Show that if $E$ and $F$ are globally generated coherent sheaves on $X$, then $E \otimes F$ is globally generated.
2) Show that if $L$ and $M$ are invertible sheaves on $X$, with $L$ ample and $M$ globally generated, then $L \otimes M$ is ample.
3) Show that if $L$ and $M$ are ample invertible sheaves on $X$, then $L \otimes M$ is ample.

**Problem 2.** Let $A$ be a Noetherian ring and $X$ a proper scheme over $\text{Spec}(A)$. Suppose that $L$ is an invertible sheaf on $X$.

1) Show that $L$ is ample if and only if its restriction $L|_{X_{\text{red}}}$ to $X_{\text{red}}$ is ample.
2) Show that if $X_1, \ldots, X_r$ are the irreducible components of $X$ (with the reduced scheme structure), then $L$ is ample if and only if $L|_{X_i}$ is ample for every $i$.

**Problem 3.** Let $X$ be a scheme of finite type over an algebraically closed field $k$. Suppose that $f: X \rightarrow \mathbb{P}^n$ is defined by the invertible sheaf $L$ on $X$ and by the sections $s_0, \ldots, s_n \in \Gamma(X, L)$.

1) A closed subscheme $V$ of $\mathbb{P}^n$ is called *nondenerate* if there is no hyperplane $H$ in $\mathbb{P}^n$ such that $V$ is a subscheme of $H$. Show that the scheme-theoretic image of $f$ is nondenerate if and only if the sections $s_0, \ldots, s_n$ are linearly independent.
2) A closed subscheme $V$ of $\mathbb{P}^n$ is called *linearly normal* if the canonical morphism

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \rightarrow H^0(V, \mathcal{O}(1)|_V)$$

is surjective. Assuming that $f$ is a closed immersion, show that $X$ is linearly normal in $\mathbb{P}^n$ if and only if $s_0, \ldots, s_n$ span $\Gamma(X, L)$.
3) Show that a nondegenerate closed subscheme $V$ of $\mathbb{P}^n$ is not linearly normal if and only if there is a nondegenerate closed subscheme $W$ of $\mathbb{P}^{n+1}$ and a point $Q$ in $\mathbb{P}^{n+1} \setminus W$ such that the projection from $Q$ induces an isomorphism $W \simeq V$.

**Problem 4.** Let $k$ be an algebraically closed field and $X \subset \mathbb{P}^n_k$ a closed subscheme defined by the ideal sheaf $\mathcal{I}_X$.

1) Show that $X$ is nondegenerate if and only if $H^0(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$, and $X$ is linearly normal if and only if $H^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$.
2) Show that if $X$ is integral and nondegenerate and $H$ is a hyperplane in $\mathbb{P}^n$, then $X \cap H$ is nondegenerate in $H \simeq \mathbb{P}^{n-1}$. 

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**Problem 5.** Let $X$ be a nonsingular projective curve over an algebraically closed field $k$. Show that if $L$ is an invertible sheaf on $X$, then $L$ is very ample if and only if for every points $P$ and $Q$ on $X$ (not necessarily distinct), we have

$$h^0(X, L(−P − Q)) = h^0(X, L) − 2.$$