Homework Set 9

Please, write solutions to **all** of the following problems. They are due on Monday April 3.

Problem 1. Show that if L is an invertible sheaf on a projective (more generally, complete) integral scheme X over the field k, such that $H^0(X,L)\neq 0$ and $H^0(X,L^{-1})\neq 0$, then $L\simeq \mathcal{O}_X$. Deduce that if $X\subseteq \mathbb{P}^n_k$ is a closed integral subscheme of positive dimension, then $H^0(X,\mathcal{O}_X(-m))=0$ for every $m>0$.

Problem 2. Let C be a nonsingular projective curve over an algebraically closed field k and let L be an invertible sheaf on C.

1) Show that if L is globally generated, then there are two sections s and t that generate L.
2) Let V be a two-dimensional subspace of $H^0(X,L)$ that generates L, and consider the canonical map
$$\phi\colon V\otimes \mathcal{O}_C\to L.$$
Show that $\ker(\phi)\simeq L^{-1}$.
3) Deduce that if L is globally generated and $H^1(C,L^{m-1})=0$, then the canonical multiplication map $H^0(C,L)\otimes H^0(C,L^m)\to H^0(C,L^{m+1})$ is surjective.

Remark. The argument in the above Problem is due to Castelnuovo and it is known as the *base-point free pencil trick*.

Problem 3. Let $X\subseteq \mathbb{P}^n_k$ be a closed subscheme defined by the ideal sheaf $\mathcal{I}_X$.

1) Show that there are hypersurfaces $H_1,\ldots,H_s$ of degree d such that X is the scheme-theoretic intersection $H_1\cap\ldots\cap H_s$ if and only if $\mathcal{I}_X(d)$ is globally generated (in this case one says that X can be cut out by hypersurfaces of degree d).
2) Let X be a *rational normal curve*, i.e. X is the image of the Veronese embedding $\mathbb{P}^1\hookrightarrow\mathbb{P}^n$ given by $(u\colon v)\to(u^n\colon u^{n-1}v\colon\ \ldots\colon v^n)$. Show that the ideal $\mathcal{I}_X$ is 2-regular, hence X can be cut out by quadrics.

Notation. If X is a projective (or more generally, complete) scheme over a field k and $\mathcal{F}$ is a coherent sheaf on X, then $h^i(\mathcal{F}):=\dim_k H^i(X,\mathcal{F})$.

Problem 4. Let C be a nonsingular projective curve over an algebraically closed field. Show that an invertible sheaf L is globally generated if and only if for every closed point P on C, we have $h^0(L(-P))=h^0(L)-1$ (note that P corresponds to an effective Cartier divisor on C and we put $L(-P):=L\otimes\mathcal{O}(-P)$).

Problem 5. Let X be a Noetherian normal integral scheme. Show that if $U\subseteq X$ is an open subset such that $\operatorname{codim}(X\smallsetminus U,X)\geq 2$, then the canonical restriction map
$$\Gamma(X,\mathcal{O}_X)\to\Gamma(U,\mathcal{O}_X)$$

is an isomorphism. Deduce that if $\mathcal{E}$ is a locally free sheaf on X, then the restriction morphism

$$\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E})$$

is an isomorphism.