Problem session 10

Problem 1. Let $G(r, n + 1)$ be the Grassmannian of $(r-1)$-dimensional linear spaces in $\mathbb{P}^n$.

1) Show that the incidence correspondence 
$$
\Gamma := \{(p, L) \in \mathbb{P}^n \times G(r, n + 1) \mid p \in L\}
$$
is a closed subset of $\mathbb{P}^n \times G(r, n + 1)$.

2) Use this to show that if $X$ is a closed subset of $\mathbb{P}^n$, then the set of $(r-1)$-dimensional linear subspaces of $\mathbb{P}^n$ that intersect $X$ non-trivially is a closed subset of $G(r, n + 1)$.

3) Let $p : \Gamma \to \mathbb{P}^n$ and $q : \Gamma \to G(r, n + 1)$ be the morphisms induced by the two projections. Show that $\mathbb{P}^n$ can be covered by open subsets $U_i$, such that $p^{-1}(U_i) \simeq U_i \times G(r-1, n)$ (over $U_i$). Similarly, $G(r, n + 1)$ can be covered by open subsets $V_i$, such that $q^{-1}(V_i) \simeq \mathbb{P}^{r-1} \times V_i$ (over $V_i$).

4) In particular, deduce that the two maps $p$ and $q$ are open.

Problem 2. Let $X$ and $Y$ be two disjoint closed subsets of $\mathbb{P}^n$. The join of $X$ and $Y$ is the union $J(X, Y)$ of all lines $pq$ in $\mathbb{P}^n$, where $p \in X$ and $q \in Y$. Show that $J(X, Y)$ is a closed subset of $\mathbb{P}^n$.

Problem 3. Let $X$ be a closed subset of $\mathbb{P}^n$. The Fano variety of lines on $X$ consists of the lines $\ell \in G(2, n + 1)$ such that $\ell \subseteq X$. Show that this is a closed subset of $G(2, n + 1)$.

Can you describe the Fano variety of lines for the quadric $xy - zw = 0$ in $\mathbb{P}^3$?

Problem 4. Let $V$ be an $n$-dimensional vector space. A complete flag in $V$ is a sequence of vector subspaces of $V$
$$V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$
with $\dim_k(V_i) = i$. Show that there is a closed subset of $\prod_{i=1}^n G(i, V)$ that parametrizes the complete flags in $V$. This is the (complete) flag variety $\text{Fl}(V)$ of $V$. 

1