Problem session 3

Problem 1. Let \( X \subseteq \mathbb{A}^n \) be a closed algebraic subset, and let \( f \in \mathcal{O}(X) \). When is the principal affine open subset \( D_X(f) \) equal to \( X \)?

Problem 2. Suppose that \( \text{char}(k) = p > 0 \), and consider the map \( f: \mathbb{A}^n \to \mathbb{A}^n \) given by \( f(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p) \).

i) Show that \( f \) is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism.

ii) Show that if \( Y \) is a closed subset of \( \mathbb{A}^n \) defined by equations with coefficients in \( \mathbb{F}_p \), then \( f \) induces a morphism from \( Y \) to \( Y \).

Problem 3. Let \( Y \subseteq \mathbb{A}^2 \) be the cuspidal curve defined by the equation \( x^2 - y^3 = 0 \). Construct a bijective morphism \( f: \mathbb{A}^1 \to Y \). Is it an isomorphism?

Problem 4.

i) Show that \( \mathbb{A}^1 \setminus \{0\} \) is an affine variety.

ii) Let \( U = \mathbb{A}^2 \setminus \{(0,0)\} \). What is \( \mathcal{O}(U) \) ?

iii) Deduce that \( U \) is not an affine variety.