Problem session 7

Problem 1. Let $X$ be a prevariety, and $f \in \mathcal{O}(X)$ a regular function on $X$. Put $X_f := \{ x \in X \mid f(x) \neq 0 \}$.

1) Show that the restriction map $\mathcal{O}(X) \to \mathcal{O}(X_f)$ induces a ring homomorphism $\rho: \mathcal{O}(X)_f \to \mathcal{O}(X_f)$, where $\mathcal{O}(X)_f$ is the ring of fractions of $\mathcal{O}(X)$ with denominators powers of $f$.
2) Show that $\rho$ is an isomorphism. (Hint: you can use the case when $X$ is affine which we proved in class; cover $X$ by affine open subsets to reduce to this case; prove first injectivity of the map, then you can use this to prove surjectivity).

Use the above problem to prove the following criterion for a prevariety to be affine.

Problem 2. Suppose that $X$ is a prevariety, and $f_1, \ldots, f_r \in \mathcal{O}(X)$ are such that

1) Each $X_{f_i}$ is an affine variety.
2) The ideal generated by $f_1, \ldots, f_r$ in $\mathcal{O}(X)$ is equal to $\mathcal{O}(X)$.

Show that $X$ is an affine variety.

The following problem considers the Segre embedding to show that the product of two projective varieties is again projective.

Problem 3. Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1)-1$, and we denote the coordinates on $\mathbb{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbb{A}^{m+1} \times \mathbb{A}^{n+1} \to \mathbb{A}^{N+1}$ given by

$$(x_i, y_j) \mapsto (x_i y_j)_{i,j}$$

induces a morphism $\phi_{m,n}: \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$.
2) Consider the ring homomorphism $f_{m,n}: k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, given by $f_{m,n}(z_{i,j}) = x_i y_j$. Show that ker($f_{m,n}$) is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if $X$ and $Y$ are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.