Problem session 1

Problem 1. Let $A$ be a ring and $m$ a positive integer. Show that the following can be endowed with group scheme structures over $Y = \text{Spec } A$:

$$X_1 = \text{Spec } k[x], \ X_2 = \text{Spec } k[x, x^{-1}], \ X_3 = \text{Spec } k[x]/(x^m - 1).$$

Describe their functors of points.

Problem 2. Describe the points of the following schemes: $\text{Spec } \mathbb{Q}[x]$, $\text{Spec } \mathbb{Z}$ and $\text{Spec } \mathbb{Z}[x]$.

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Problem 3. Show that every affine scheme is quasi-compact (as a topological space). Deduce that a scheme is quasi-compact if and only if it can be covered by finitely many affine open subsets.

Problem 4. Let $X$ be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$. For every $x$ in $X$ we denote by $f_x$ the image of $f$ in the stalk $\mathcal{O}_{X,x}$, and by $m_x \subset \mathcal{O}_{X,x}$ the maximal ideal.

i) Show that the set

$$X_f = \{ x \in X \mid f_x \notin m_x \}$$

is open (this set should be interpreted as “the set where $f$ does not vanish”). Describe this set when $X = \text{Spec } A$.

ii) Show that the restriction map induces a ring homomorphism

$$\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_X).$$

Show that if $X$ is quasi-compact, then this is injective.

iii) If $X$ can be covered by finitely many affine open subsets $U_1, \ldots, U_n$ such that all $U_i \cap U_j$ are quasi-compact, then the above morphism is an isomorphism.

Problem 5. For an arbitrary scheme $X$, use the canonical morphism

$$X \to \text{Spec } \Gamma(X, \mathcal{O}_X)$$

to prove the following criterion for $X$ to be affine: if $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ are such that they generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$ and all $X_{f_i}$ are affine schemes, then $X$ is affine.