Problem session 1

Problem 1. Let A be a ring and m a positive integer. Show that the following can be endowed with group scheme structures over $Y = \operatorname{Spec} A$:

$$X_1 = \operatorname{Spec} k[x],\ X_2 = \operatorname{Spec} k[x, x^{-1}],\ X_3 = \operatorname{Spec} k[x]/(x^m - 1).$$

Describe their functors of points.

Problem 2. Describe the points of the following schemes: $\operatorname{Spec} \mathbb{Q}[x]$, $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} \mathbb{Z}[x]$.

Problem 2. Let A be a ring and m a positive integer. Show that the following can be endowed with group scheme structures over $Y = \operatorname{Spec} A$:

$$X_1 = \operatorname{Spec} k[x],\ X_2 = \operatorname{Spec} k[x, x^{-1}],\ X_3 = \operatorname{Spec} k[x]/(x^m - 1).$$

Describe their functors of points.

Problem 3. Show that every affine scheme is quasi-compact (as a topological space). Deduce that a scheme is quasi-compact if and only if it can be covered by finitely many affine open subsets.

Problem 4. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$. For every x in X we denote by f_x the image of f in the stalk $\mathcal{O}_{X,x}$, and by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal.

i) Show that the set
$$X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x\}$$
is open (this set should be interpreted as "the set where f does not vanish"). Describe this set when $X = \operatorname{Spec} A$.

ii) Show that the restriction map induces a ring homomorphism
$$\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_X).$$
Show that if X is quasi-compact, then this is injective.

iii) If X can be covered by finitely many affine open subsets $U_1, \ldots, U_n$ such that all $U_i \cap U_j$ are quasi-compact, then the above morphism is an isomorphism.

Problem 5. For an arbitrary scheme X, use the canonical morphism

$$X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$$

to prove the following criterion for X to be affine: if $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ are such that they generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$ and all X_{f_i} are affine schemes, then X is affine.