The first problem requires the use of the Tor functors. We recall first their definition and the basic properties. Suppose that \( A \) is a commutative ring and \( M \) is an \( A \)-module. The functor \( M \otimes_A - \) is right exact. The category of \( A \)-modules has enough projective objects and therefore we can construct the left derived functors of the above functor. The \( i \)th derived functor is denoted by \( \text{Tor}^A_i(M, -) \).

It follows by definition that \( \text{Tor}^A_0(M, N) \simeq M \otimes_A N \) and that if we have an exact sequence of \( A \)-modules

\[
0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0
\]

then we get a long exact sequence

\[
\cdots \rightarrow \text{Tor}^A_i(M, N') \rightarrow \text{Tor}^A_i(M, N) \rightarrow \text{Tor}^A_i(M, N'') \rightarrow \text{Tor}^A_{i-1}(M, N') \rightarrow \cdots
\]

It is an easy exercise to deduce from this and from the definition of flatness that \( M \) is flat if and only if \( \text{Tor}^A_i(M, N) = 0 \) for every \( N \) and for every \( i \geq 1 \). Moreover, it is enough to have this for \( i = 1 \) and every \( N \).

The fact that the tensor product commutes with arbitrary direct sums implies that the same remains true for the Tor functors. A slightly trickier result is that we have \( \text{Tor}^A_i(M, N) \simeq \text{Tor}^A_i(N, M) \) for every \( i, M \) and \( N \). This follows using the commutativity of the tensor product and by computing Tor using free resolutions for both \( M \) and \( N \) at the same time.

The purpose of the first problem is to prove a version of the local flatness criterion.

**Problem 1.** Let \( A \) be a ring and \( I \) an ideal of \( A \) that is nilpotent, i.e. there is \( q \) such that \( I^q = 0 \). If \( M \) is an \( A \)-module, then the following are equivalent:

1. \( M \) is flat over \( A \).
2. \( M/IM \) is flat over \( A/I \) and the canonical morphism \( I \otimes M \rightarrow M \) is injective.

For the implication \((2) \Rightarrow (1)\), suppose that the condition in \((2)\) is satisfied.

(a) Show that \( \text{Tor}_1(M, A/I) = 0 \).
(b) Deduce that for every \( A/I \)-module \( N \) we have \( \text{Tor}_1^A(M, N) = 0 \) (hint: consider a free presentation of \( N \)).
(c) Prove now by induction on \( m \geq 1 \) that if \( N \) is an \( A \)-module annihilated by \( I^m \), then \( \text{Tor}_1^A(M, N) = 0 \).

By taking \( m = q \), we deduce that \( M \) is flat over \( A \).

**Remark.** There is another version of the local flatness criterion: if \( (A, m) \rightarrow (B, n) \) is a local morphism of local Noetherian rings and if \( M \) is a finitely generated \( B \)-module, then the equivalence \((1) \Leftrightarrow (2)\) still holds if \( I \subseteq m \). Moreover, note that if we take \( I = m \), then \( M/IM \) is automatically flat over \( A/I \). For a proof, see Matsumura’s book.
We will use this to describe infinitesimal deformations of a scheme. Let $X$ be a scheme of finite type over an algebraically closed field $k$. An infinitesimal deformation of $X$ (over $A$) is a Cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \quad g \\
\text{Spec}(k) & \longrightarrow & \text{Spec}(A)
\end{array}
$$

with $g$ flat and of finite type and where $A$ is a finite local $k$-algebra (i.e. $\text{Spec}(A)$ is supported at one point).

In fact, we will be interested in embedded deformations. Suppose that $X$ is a closed subscheme of a scheme $Y$ of finite type over $k$ and that $A$ is as above. An infinitesimal embedded deformation of $X$ (over $A$) is a closed subscheme $X \hookrightarrow Y \times \text{Spec}(A)$ that is flat over $\text{Spec}(A)$ and such that $X \times_{\text{Spec}(A)} \text{Spec}(k) = X$ as closed subschemes of $Y$.

Suppose that $\phi: A' \rightarrow A$ is a morphism of local finite $k$-algebras inducing $f: \text{Spec}(A) \hookrightarrow \text{Spec}(A')$. If $X \hookrightarrow Y \times \text{Spec}(A')$ is an embedded deformation of $X$ over $A'$ then $f^*X = X$.

Problem 2. With the above notation, suppose that $Y = \text{Spec}(R)$ is affine and that $X$ is defined by the ideal $J \subseteq R \otimes_k A$. Let $X'$ be a closed subscheme of $Y \times \text{Spec}(A')$ defined by the ideal $J' \subseteq R \otimes_k A'$ such that $J' \cdot (R \otimes_k A) = J$. Show that $X'$ is flat over $\text{Spec}(A')$ if and only if the following two conditions hold for some (every) system of generators $f_1, \ldots, f_r$ of $J$:

(a) We have liftings $f'_i$ of the $f_i$ to $J'$ that generate $J'$ (in fact every such liftings to $J'$ will generate $J'$).

(b) For every relation $\sum_i g_i f_i = 0$ with the $g_i$ in $R \otimes_k A$ there are liftings $g'_i$ of the $g_i$ to $R \otimes_k A'$ such that $\sum_i g'_i f'_i = 0$.

In the next problem we use the above description of liftings to study the first-order embedded deformations of $X$.

Problem 3. Let $X$ be a closed subscheme of $Y$. Our goal is to describe the set of embedded deformations of $X$ over $A = k[t]/(t^2)$.

(a) Suppose first that $Y = \text{Spec}(R)$ is affine and let $f_1, \ldots, f_r$ be generators of the ideal $I$ of $X$ in $Y$. Show that the ideal of every embedded deformation of $X$ over $k[t]/(t^2)$ is generated by elements of the form $f_i + tg_i$ for suitable elements $g_i \in R$ with the property that the morphism $R' \rightarrow R$ taking $e_i$ to $g_i$ induces a morphism
of $R$-modules $I \to R/I$ (note that we have also a surjection $R^e \to I$ that takes each $e_i$ to $f_i$).

(b) Show that two sets $\{g_i\}_i$ and $\{g'_i\}_i$ as above define the same closed subscheme of $Y \times \text{Spec}(k[t]/(t^2))$ if and only if the corresponding morphisms $I \to R/I$ are the same.

(c) Deduce that for an arbitrary $Y$ (not necessarily affine), if $\mathcal{I}$ is the sheaf of ideals defining $X$, then the set of embedded deformations of $X$ over $k[t]/(t^2)$ is isomorphic to the space of global sections of the normal sheaf of $X$ in $Y$, namely to $\text{Hom}_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$. 